The nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions

The nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions

S Belliard and E Ragoucy

Laboratoire de Physique Théorique LAPTH UMR 5108 du CNRS, associée à l'Université de Savoie BP 110, F-74941 Annecy-le-Vieux Cedex, France

E-mail: samuel.belliard@lapp.in2p3.fr and eric.ragoucy@lapp.in2p3.fr

Abstract

We present in a unified and detailed way the nested Bethe ansatz for open spin chains based on $\mathcal{Y}(gl(\mathfrak{n}))$, $\mathcal{Y}(gl(\mathfrak{m}|\mathfrak{n}))$, $\widehat{\mathcal{U}}_q(gl(\mathfrak{n}))$ or $\widehat{\mathcal{U}}_q(gl(\mathfrak{m}|\mathfrak{n}))$ (super)algebras, with arbitrary representations (i.e., 'spins') on each site of the chain and diagonal boundary matrices ($K^+(u)$, $K^-(u)$). The nested Bethe ansatz applies for a general $K^-(u)$, but a particular form of the $K^+(u)$ matrix. The construction extends and unifies the results already obtained for open spin chains based on the fundamental representation and for some particular super-spin chains. We give the eigenvalues, Bethe equations and the explicit form of the Bethe vectors for the corresponding models. The Bethe vectors are also expressed using a trace formula.

1. Introduction

The systematic studies of the open spins chains using the R matrices formalism start with the seminal papers of Cherednik [1] and Sklyanin [2], who generalized to these models the QISM approach developed by the Leningrad school. They introduced the reflection algebra as the fundamental ingredient to construct the Abelian Bethe subalgebra and ensure integrability of the model. This algebra is a subalgebra of the FRT algebra introduced by Leningrad group for the periodic spin chains (for a review, see e.g. [3] and references therein). The boundary conditions are encoded in two matrices, $K^-(u)$ solution of the reflection equation, see equation (3.8) below, and $K^+(u)$ solution of the dual equation, see equation (3.14). With these matrices and the standard closed spin chain monodromy matrix, one can construct a transfer matrix that belongs to the Bethe subalgebra. The existence of this subalgebra leads to the integrability of the model when the expansion of the transfer matrix as a series provides a sufficient number of operators in involution. In the following, we consider that this number is sufficient.

After proving integrability of the model, the next step is to find the eigenvalues and eigenvectors of this Bethe subalgebra. It depends on the choice of the boundary matrices. Focusing on diagonalizable boundary matrices, two main cases can be distinguished: $K^+(u)$ and $K^-(v)$ are diagonalizable in the same basis; or not.

Very little is known in the latter case, apart from two recent approaches developed for the *XXZ* spin chain and that do not rely on the (nested) algebraic Bethe ansatz [4–6]: in [7], the reflection equation is replaced by a deformed Onsager algebra (which may be another presentation of the reflection algebra); and in [8], eigenvalues are computed using generalized TQ relations when $K^-(u)$ and $K^+(u)$ obey some relations, or when the deformation parameter is root of unity.

The first case can be divided into two sub-families, depending whether (i) the diagonalization matrix is a constant or (ii) depends on the spectral parameter. Again, in the case (ii), only some results are known from the gauge transformation construction of [9–11], that allow us to relate non-diagonal solutions to diagonal ones via a Face–vertex correspondence. Case (i) is the one studied by the analytical Bethe ansatz [12] and corresponds to diagonal matrices

$$K^-(u) = \mathrm{diag}(\underbrace{k^-(u), \ldots, k^-(u)}_{a_-}, \underbrace{\bar{k}^-(u), \ldots, \bar{k}^-(u)}_{\mathfrak{m}+\mathfrak{n}-a_-}), \tag{1.1}$$

$$K^+(u) = \mathrm{diag}(\underbrace{k^+(u), \ldots, k^+(u)}_{a_+}, \underbrace{\bar{k}^+(u), \ldots, \bar{k}^+(u)}_{\mathfrak{m}+\mathfrak{n}-a_+}). \tag{1.2}$$

Indeed, using this ansatz, eigenvalues of the transfer matrix can be computed for all (open or closed) chains based on $gl(\mathfrak{n})$ and $gl(\mathfrak{m}|\mathfrak{n})$ (super)algebras and their deformation, and with arbitrary representations on each site [13–15]. It is in general believed that a nested (algebraic) Bethe ansatz (NBA) can provide the eigenvectors of the corresponding models. This was shown in a unified way in [16] for closed spin chains.

We present here the open spin chains case. We will show that the standard NBA approach does <u>not</u> work in the general case. Keeping a general diagonal solution for $K^-(u)$, one needs to take $a_+ = 0$ or $a_+ = 1$ to perform a complete nested algebraic Bethe ansatz. In this case, the couple $(K^-(u), K^+(u))$ will be called a *NABA couple*. When one studies an open spin chain possessing a couple of diagonal matrices $(K^-(u), K^+(u))$ that is not of type NABA, one can start the first step of the NBA approach, but then needs to switch (and end) the calculation with an analytical Bethe ansatz, as has been done in e.g. [10, 11, 17].

In the present paper, we focus on NABA couples. Performing NBA, we compute the Bethe ansatz equations, the eigenvalues and the eigenvectors of the corresponding transfer matrix and show where the constraint $a_+ = 0$ or $a_+ = 1$ is needed in the calculation. Our presentation considers *universal transfer matrices* in the sense that the calculation applies to transfer matrices based on $gl(\mathfrak{n})$ and $gl(\mathfrak{m}|\mathfrak{n})$ algebras and their deformation, with any finite-dimensional irreducible representations of the monodromy matrix. In particular, it encompasses the preview results obtained for fundamental representations [10, 17, 18].

In addition to the derivation of the Bethe ansatz equations and transfer matrix spectrum, our main result is the explicit construction of the Bethe vectors. This is reflected in e.g. the trace formula (see theorem 7.1 at the end of the paper).

The plan of the paper is as follows. In section 2, we introduce the different notations and *R*-matrices we use in the paper. Then, in section 3, we present, using the FRT [19] formalism, the algebras concerned with our approach. They are generalizations of loop algebras (quantum algebras or Yangians, and their graded versions) and denoted $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$. They contain as subalgebras the reflection algebras, denoted $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$. We also construct mappings

$$\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}} \to \mathfrak{D}_{\mathfrak{m}-1|\mathfrak{n}} \to \cdots \to \mathfrak{D}_{1|1} \quad \text{or} \quad \mathfrak{D}_2$$

that are needed for the nesting. In section 4, we present the finite-dimensional irreducible representations of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ and we compute the form of $T^{-1}(u)$. We also construct the representations of $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ from the $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ ones. In section 5, as a warm up, we recall the algebraic Bethe ansatz, which deals with spin chains based on $gl(2)$, $gl(1|1)$ algebras and their quantum deformations. Then, in section 6, we perform the nested Bethe ansatz in a very detailed and pedestrian way and up to the end. Finally, in section 7, we study the Bethe vectors that have been constructed in the previous section, showing connection with a *trace formula*. As a conclusion, we discuss our results and present some possible applications or extensions of our work.

2. Notations

2.1. Graded auxiliary spaces

We use the so-called auxiliary space framework. In this formalism, one deals with the multiple tensor product of vectorial spaces $\mathcal{V} \otimes \cdots \otimes \mathcal{V}$, and operators (defining an algebra $\mathcal{A}$) therein. For any matrix-valued operator, $A := \sum_{ij} E_{ij} \otimes a_{ij} \in \mathrm{End}(\mathcal{V}) \otimes \mathcal{A}$, we set

$$A_k := \sum_{ij} \mathbb{I}^{\otimes(k-1)} \otimes E_{ij} \otimes \mathbb{I}^{\otimes(m-k)} \otimes a_{ij} \in \mathrm{End}(\mathcal{V}^{\otimes m}) \otimes \mathcal{A}, \qquad 1 \leqslant k \leqslant m, \qquad (2.1)$$

where E_{ij} are elementary matrices, with 1 at position (i, j) and 0 elsewhere. The notation is valid for complex matrices, taking $\mathcal{A} := \mathbb{C}$ and using the isomorphism $\mathrm{End}(\mathcal{V}) \otimes \mathbb{C} \sim \mathrm{End}(\mathcal{V})$.

We will work on $\mathbb{Z}_2$-graded spaces $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$. The elementary $\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ column vectors e_i (with 1 at position i and 0 elsewhere) and elementary $\mathrm{End}(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})$ matrices E_{ij} have grade:

$$[e_i] = [i] \qquad \text{and} \qquad [E_{ij}] = [i] + [j]. \qquad (2.2)$$

This grading is also extended to the superalgebras we deal with, see section 3.1 below. The tensor product is graded accordingly:

$$(a_{ij} \otimes a_{kl})(a_{pq} \otimes a_{rs}) = (-1)^{([k]+[l])([p]+[q])}(a_{ij}a_{pq} \otimes a_{kl}a_{rs}). \qquad (2.3)$$

The transposition $(.)^t$ and trace $\mathrm{str}(.)$ operators are also graded:

$$A^t = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{[j]+[j][i]} E_{ji} \otimes a_{ij}, \qquad \mathrm{str}\, A = \sum_{i=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{[i]} a_{ii} \quad \text{for} \quad A = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes a_{ij}. \qquad (2.4)$$

To simplify the presentation we work with the *distinguished* $\mathbb{Z}_2$-grade defined by

$$[i] = \begin{cases} 0, & 1 \leqslant i \leqslant \mathfrak{m}, \\ 1, & \mathfrak{m} + 1 \leqslant i \leqslant \mathfrak{m} + \mathfrak{n}. \end{cases} \qquad (2.5)$$

Simplification in the expressions follows from the rule $[i][j] = [i]$ when $i \leqslant j$, which is valid only for the distinguished grade. The non-graded case is recovered setting $\mathfrak{n} = 0$ and $[k] = 0$.

2.2. Spectral parameter transformations

For spectral parameter u we use the following notations:

$$\iota(u) = \begin{cases} -u & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ \frac{1}{u} & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}); \end{cases} \qquad \widetilde{u} = \begin{cases} u - \frac{(\mathfrak{m}-\mathfrak{n})\hbar}{2} & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ uq^{\frac{(\mathfrak{m}-\mathfrak{n})}{2}} & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}); \end{cases}$$

$$u^{(k)} = \begin{cases} u + \frac{\hbar}{2}(-1)^{[k]} & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ uq^{-\frac{1}{2}+[k]} & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}); \end{cases} \qquad u^{\overline{(k)}} = \begin{cases} u - \frac{\hbar}{2}(-1)^{[k]} & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ uq^{\frac{1}{2}-[k]} & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}); \end{cases}$$

$$u^{(k\ldots l)} = (\cdots (u^{(k)})^{(k+1)} \cdots)^{(l)}.$$

2.3. R-matrices

In what follows, we will deal with different types of matrices $R \in \text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V})$, all obeying a (graded) Yang–Baxter equation (written in auxiliary space $\text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V}) \otimes \text{End}(\mathcal{V})$):

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{2.6}$$

The *R*-matrix satisfies the unitarity relation,

$$R_{12}(u, v)R_{21}(v, u) = \zeta(u, v)\mathbb{I} \otimes \mathbb{I}, \tag{2.7}$$

and crossing unitarity (see (2.11) below for the definition of $\bar{R}$),

$$R_{12}^{t_1}(u, v)M_1\bar{R}_{12}^{t_2}(\iota(\widetilde{v}), \iota(\widetilde{u}))M_1^{-1} = \overline{\zeta}(u, v)\mathbb{I} \otimes \mathbb{I}, \tag{2.8}$$

where $\zeta(u, v)$ and $\overline{\zeta}(u, v)$ are $\mathbb{C}$-functions depending on the model under consideration, M is a $\mathbb{C}$-valued matrix defined in appendix A for each model and t_a is the transposition in the auxiliary space a. All the *R*-matrices used here also obey the parity relation:

$$R_{12}(u, v)^{t_1 t_2} = R_{21}(u, v). \tag{2.9}$$

To each *R*-matrix, one associates an algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ using the FRT formalism. Below, we focus on infinite-dimensional associative algebras based on $gl(\mathfrak{n})$ and $gl(\mathfrak{m}|\mathfrak{n})$ Lie (super)algebras and their *q*-deformation. We denote these algebras $\mathcal{A}_{\mathfrak{n}} = Y(\mathfrak{n})$ or $\widehat{\mathcal{U}}_q(\mathfrak{n})$ and $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}} = \mathcal{Y}(\mathfrak{m}|\mathfrak{n})$ or $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$. We will write also $\mathcal{A}_{\mathfrak{m}|0} = \mathcal{A}_{\mathfrak{m}}$. We will encompass all *R*-matrices of these algebras writing

$$R_{12}(u, v) = \mathfrak{b}(u, v)\mathbb{I} \otimes \mathbb{I} + \sum_{i, j=1}^{\mathfrak{m}+\mathfrak{n}} \mathfrak{w}_{ij}(u, v)E_{ij} \otimes E_{ji}. \tag{2.10}$$

All functions are defined in appendix B for each case (one can refer to [16] for details and references). To define the reflection equation, we need another *R*-matrix:

$$\bar{R}_{12}(u, v) = R_{12}(u, \iota(v)) = \bar{\mathfrak{b}}(u, v)\mathbb{I} \otimes \mathbb{I} + \sum_{i, j=1}^{\mathfrak{m}+\mathfrak{n}} \bar{\mathfrak{w}}_{ij}(u, v)E_{ij} \otimes E_{ji}. \tag{2.11}$$

From (2.6) we can deduce the relation between these two *R* matrices:

$$R_{12}(u_1, u_2)\bar{R}_{13}(u_1, u_3)\bar{R}_{23}(u_2, u_3) = \bar{R}_{23}(u_2, u_3)\bar{R}_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{2.12}$$

We will also use 'reduced' *R*-matrices $R^{(k)}(u)$, deduced from $R(u)$ by suppressing all the terms containing indices j with $j < k$:

$$R_{12}^{(k)}(u, v) = (\mathbb{I}^{(k)} \otimes \mathbb{I}^{(k)})R_{12}(u, v)(\mathbb{I}^{(k)} \otimes \mathbb{I}^{(k)})$$

$$= \mathfrak{b}(u, v)\mathbb{I}^{(k)} \otimes \mathbb{I}^{(k)} + \sum_{i, j=k}^{\mathfrak{m}+\mathfrak{n}} \mathfrak{w}_{ij}(u, v)E_{ij} \otimes E_{ji}, \tag{2.13}$$

where $\quad \mathbb{I}^{(k)} = \sum_{i=k}^{\mathfrak{m}+\mathfrak{n}} E_{ii}, \quad \forall k. \tag{2.14}$

$R_{12}^{(k)}(u, v)$ corresponds to the *R*-matrix of[1] $\mathcal{A}_{\mathfrak{m}+1-k|\mathfrak{n}}$. We will also use

$$R_{12}^{(k, p)}(u, v) = (\mathbb{I}^{(k)} \otimes \mathbb{I}^{(p)})R_{12}(u, v)(\mathbb{I}^{(k)} \otimes \mathbb{I}^{(p)})$$

$$= \mathfrak{b}(u, v)\mathbb{I}^{(k)} \otimes \mathbb{I}^{(p)} + \sum_{i, j=\max(k, p)}^{\mathfrak{m}+\mathfrak{n}} \mathfrak{w}_{ij}(u, v)E_{ij} \otimes E_{ji}. \tag{2.15}$$

---

[1] We will write, for a generic k, $\mathfrak{m} - k|\mathfrak{n}$, keeping in mind that one should write $0|\mathfrak{n} - (k - \mathfrak{m})$ when $k > \mathfrak{m}$.

Note that $R_{12}^{(k,k)}(u, v) = R_{12}^{(k)}(u, v)$. We define the 'normalized reduced' R-matrices:

$$\mathbb{R}_{12}^{(k,p)}(u, v) = \frac{1}{\mathfrak{a}_p(u, v)} R_{12}^{(k,p)}(u, v) \qquad \text{with} \quad \mathbb{R}_{12}^{(k,k)}(u, v)\mathbb{R}_{21}^{(k,k)}(v, u) = \mathbb{I}^{(k)} \otimes \mathbb{I}^{(k)}. \quad (2.16)$$

3. Algebraic structures

3.1. FRT formalism

The FRT (or RTT) relations [19–21] allow us to generate all the relations between the generators of the graded unital associative algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$. We gather the $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ generators into a $(\mathfrak{m} + \mathfrak{n}) \times (\mathfrak{m} + \mathfrak{n})$ matrix acting in an auxiliary space $\mathcal{V} = \mathbb{C}^{\mathfrak{m}|\mathfrak{n}}$ whose entries are a formal series of a complex parameter u:

$$T(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t_{ij}(u) \in \text{End}(\mathcal{V}) \otimes \mathcal{A}[[u, u^{-1}]]. \quad (3.1)$$

Since the auxiliary space $\text{End}(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})$ is interpreted as a representation of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, the $\mathbb{Z}_2$-grading of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ must correspond to the one defined on $\text{End}(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}})$ matrices (see section 2). Hence, the generator $t_{ij}(u)$ has grade $[i] + [j]$, so that the monodromy matrix $T(u)$ is globally even. As for matrices, the tensor product of algebras will be graded, as well as between algebras and matrices,

$$(E_{ij} \otimes t_{ij}(u))(E_{kl} \otimes t_{kl}(v)) = (-1)^{([i]+[j])([k]+[l])} E_{ij} E_{kl} \otimes t_{ij}(u) t_{kl}(v). \quad (3.2)$$

The 'real' generators $t_{ij}^{(n)}$ of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ appear upon expansion of $t_{ij}(u)$ in u. For the (super) Yangians $\mathcal{Y}(\mathfrak{n})$ and $\mathcal{Y}(\mathfrak{m}|\mathfrak{n})$, $t_{ij}(u)$ is a series in u^{-1}:

$$t_{ij}(u) = \sum_{n=0}^{\infty} t_{ij}^{(n)} u^{-n} \qquad \text{with} \qquad t_{ij}^{(0)} = \delta_{ij}. \quad (3.3)$$

In the quantum affine (super)algebra [22, 23] without central charge case, a complete description of the algebras requires the introduction of two matrices $L^{\pm}(u)$. However, in the context of evaluation representations it is sufficient to consider only $T(u) = L^{+}(u)$ to construct a transfer matrix. Indeed, in an evaluation representation, the choices $T(u) = L^{-}(u)$ or $T(u) = L^{+}(u) - L^{-}(u)$ lead to the same operator up to a multiplication function. Then, the RTT relations take the form:

$$R_{12}(u, v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u, v). \quad (3.4)$$

$\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ has the following antimorphisms:

Matrix inversion   inv: $T(u) \rightarrow T^{-1}(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t'_{ij}(u),$

matrix transposition t: $T(u) \rightarrow T^t(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} (E_{ij})^t \otimes t_{ij}(u),$ (3.5)

spectral parameter inversion ι: $T(u) \rightarrow T(\iota(u)).$

$\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ has a Hopf algebra structure, with coproduct

$$\Delta(T(u)) = T(u) \dot{\otimes} T(u) = \sum_{i,j,k=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{([k]+[i])([k]+[j])} E_{ij} \otimes t_{ik}(u) \otimes t_{kj}(u). \quad (3.6)$$

More generally, one defines recursively for $L \geqslant 2$, the algebra homomorphism

$$\Delta^{(L+1)} = (\text{id}^{\otimes(L-1)} \otimes \Delta) \circ \Delta^{(L)} \qquad \text{with} \quad \Delta^{(2)} = \Delta \quad \text{and} \quad \Delta^{(1)} = \text{id}. \quad (3.7)$$

3.2. Reflection algebra and $K(u)$ matrices

The $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ algebra is enough to construct a transfer matrix leading to periodic spin chain models. In the context of open spin chains, one needs another algebra, the reflection algebra $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$, which turns out to be a subalgebra of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$. Indeed, physically, one can interpret the FRT relation as encoding the interaction between the spins of the chain. Hence, it is the only relation needed to describe a periodic chain. On the other hand, in the case of an open chain, the interaction with the boundaries has to be taken into account. Following the seminal paper of Sklyanin [2], we construct the reflection algebra and the dual reflection equation for the boundary scalar matrices $K^-(u)$ and $K^+(u)$. We first define the matrix $K^-(u)$ to be the solution of the reflection equation:

$$R_{12}(u_1, u_2)K_1^-(u_1)\bar{R}_{21}(u_1, u_2)K_2^-(u_2) = K_2^-(u_2)\bar{R}_{12}(u_1, u_2)K_1^-(u_1)R_{21}(u_1, u_2). \tag{3.8}$$

Depending on the type of R-matrix one considers, solutions to the reflection equation have been classified: see [24] for the Yangian and super-Yangian cases; in the other cases, partial classifications have been obtained in e.g. [17, 25]. In all cases, diagonal solutions of the reflection equations are known. They take the form (up to normalization),

$$K^-(u) = \begin{cases} \mathrm{diag}(\underbrace{u - c_-, \ldots, u - c_-}_{a}, \underbrace{-u - c_-, \ldots, -u - c_-}_{\mathfrak{m}+\mathfrak{n}-a}) & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}), \\ \mathrm{diag}(\underbrace{u^2 - c_-^2, \ldots, u^2 - c_-^2}_{a}, \underbrace{u^{-2} - c_-^2, \ldots, u^{-2} - c_-^2}_{\mathfrak{m}+\mathfrak{n}-a}) & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}), \end{cases} \tag{3.9}$$

where c_- is a free complex parameter and a is an integer such that $0 \leqslant a \leqslant \mathfrak{m} + \mathfrak{n}$. From this $K^-(u)$ matrix and the monodromy matrix $T(u)$ for closed spin chains, we can construct the monodromy matrix of the open spin chain:

$$D(u) = T(u)K^-(u)T^{-1}(\iota(u)) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes d_{ij}(u), \tag{3.10}$$

$$d_{ij}(u) = \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{([i]+[a])([a]+[j])} \kappa_a(u) t_{ia}(u) t'_{aj}(\iota(u)). \tag{3.11}$$

From (3.4) and (3.8), we can prove that $D(u)$ also satisfies the reflection equation:

$$R_{12}(u_1, u_2)D_1(u_1)\bar{R}_{21}(u_1, u_2)D_2(u_2) = D_2(u_2)\bar{R}_{12}(u_1, u_2)D_1(u_1)R_{21}(u_1, u_2). \tag{3.12}$$

This relation defines the reflection algebra $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$. The algebra $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ is a left coideal [26] of the algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ with the coproduct:

$$\Delta D_{[2]}(u) = T_{[1]}(u)D_{[2]}(u)T_{[1]}^{-1}(\iota(u)) \in \mathrm{End}(\mathcal{V}) \otimes \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \otimes \mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}, \tag{3.13}$$

where $[i]$ labels the two copies of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$. This expression allows us to increase the number of sites for an open spin chain in the same way one does for periodic ones: one acts on the monodromy matrix with the coproduct and then represents the new copy of algebra on the new 'site'.

We also need a dual equation to construct transfer matrices in involution:

$$R_{12}(u_2, u_1)\left(K_1^+(u_1)\right)^{t_1} M_1^{-1} \bar{R}_{21}(\iota(\tilde{u}_1), \iota(\tilde{u}_2)) M_1 \left(K_2^+(u_2)\right)^{t_2}$$
$$= \left(K_2^+(u_2)\right)^{t_2} M_1 \bar{R}_{12}(\iota(\tilde{u}_1), \iota(\tilde{u}_2)) M_1^{-1} \left(K_1^+(u_1)\right)^{t_1} R_{21}(u_2, u_1), \tag{3.14}$$

where M is given in appendix A. From the property

$$R_{12}(u_1, u_2)M_1 M_2 = M_1 M_2 R_{12}(u_1, u_2), \tag{3.15}$$

one can construct solutions to the dual reflection equation using $K^-(u)$ solutions:

$$(K^+(u))^t = M K^-(\iota(\widetilde{u})). \tag{3.16}$$

With $D(u)$ and $K^+(u)$ we construct the transfer matrix:

$$d(u) = \text{str}(K^+(u) D(u)). \tag{3.17}$$

The reflection equation and its dual form ensure the commutation relation $[d(u), d(v)] = 0$. Thus, $d(u)$ generates (via an expansion in u) a set of L (the number of sites) independent integrals of motion (or charges) in involution which ensure integrability of the model.

3.2.1. Commutation relations of $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$.   Projecting (3.12) on the $E_{ij} \otimes E_{kl}$ basis we get the commutation relations for $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$:

$$
\begin{aligned}
[d_{ij}(u), d_{kl}(v)\} = &-\delta_{kj} \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \frac{\bar{\mathfrak{w}}_{ka}(u,v)}{\bar{\mathfrak{b}}(u,v)} (-1)^{([i]+[l])([a]+[k])} d_{ia}(u) d_{al}(v) \\
&+ \delta_{il} \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \frac{\bar{\mathfrak{w}}_{ia}(u,v)}{\bar{\mathfrak{b}}(u,v)} (-1)^{[l]+[a][k]+[a][j]+[k][j]} d_{ka}(v) d_{aj}(u) \\
&- \delta_{ij} \frac{\mathfrak{w}_{ik}(u,v)}{\mathfrak{b}(u,v)} \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \frac{\bar{\mathfrak{w}}_{ia}(u,v)}{\bar{\mathfrak{b}}(u,v)} (-1)^{([a]+[k])([k]+[l])} d_{ka}(u) d_{al}(v) \\
&+ \delta_{ij} \frac{\mathfrak{w}_{il}(u,v)}{\mathfrak{b}(u,v)} \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \frac{\bar{\mathfrak{w}}_{ia}(u,v)}{\bar{\mathfrak{b}}(u,v)} (-1)^{([a]+[l])([k]+[l])} d_{ka}(v) d_{al}(u) \\
&- \frac{\mathfrak{w}_{ik}(u,v)}{\mathfrak{b}(u,v)} (-1)^{([i]+[k])([k]+[l])} d_{kj}(u) d_{il}(v) \\
&+ \frac{\mathfrak{w}_{jl}(u,v)}{\mathfrak{b}(u,v)} (-1)^{([i]+[l])([k]+[l])} d_{kj}(v) d_{il}(u),
\end{aligned} \tag{3.18}
$$

where $[x, y\} = xy - (-1)^{[x][y]} yx$ is the graded commutator.

3.3. Embeddings of $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ algebras

The algebraic cornerstone for the nested Bethe ansatz is a recursion relation on the $\mathfrak{D}_{m|n}$ algebraic structure. In this section, we present a coset construction for $\mathfrak{D}_{m|n}$ algebras (see theorem 3.1), that extends to the coideal property (see lemma 3.2 and theorem 3.3).

Theorem 3.1. *For $k = 1, 2, \ldots, \mathfrak{m} + \mathfrak{n} - 1$, let $F^{(k)}$ be a linear combination of elements $d_{i_1 j_1}(u_1) \cdots d_{i_l j_l}(u_l)$ with all indices $i_p, j_p > k - 1$, and let $\mathcal{I}_k$ be the left ideal generated by $d_{ij}(u)$ for $i > j$ and $j < k$. Then, we have the following properties:*

$$d_{ij}(u) F^{(k)} \equiv 0 \quad \mod \mathcal{I}_k, \qquad for \quad i > j \quad and \quad j < k, \tag{3.19}$$

$$[d_{ii}(u), F^{(k)}] \equiv 0 \quad \mod \mathcal{I}_k, \qquad for \quad i < k. \tag{3.20}$$

Using the functions ψ_j given in (B.2), we introduce the generators:

$$\widehat{D}^{(k)}(u) = \sum_{i,j=k}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes d_{ij}^{(k)}(u), \tag{3.21}$$

$$d_{ij}^{(k)}(u) = d_{ij}(u^{(1...k-1)}) - \delta_{ij} \sum_{a=1}^{k-1} q^{2(k-1-a)-4\sum_{l=a+1}^{k-1}[l]} \psi_a(u^{(a)}) d_{aa}(u^{(1...k-1)}). \tag{3.22}$$

They satisfy in $\mathfrak{D}_{\mathrm{m|n}}/\mathcal{I}_k$ the reflection equation for $\mathfrak{D}_{\mathrm{m-k+1|n}}$:

$$R_{12}^{(k)}(u_1, u_2)\widehat{D}_1^{(k)}(u_1)\bar{R}_{21}^{(k)}(u_1, u_2)\widehat{D}_2^{(k)}(u_2)$$
$$\equiv \widehat{D}_2^{(k)}(u_2)\bar{R}_{12}^{(k)}(u_1, u_2)\widehat{D}_1^{(k)}(u_1)R_{21}^{(k)}(u_1, u_2) \mod \mathcal{I}_k. \tag{3.23}$$

Proof. We first prove relation (3.20) for $k = 2$, the case $k = 1$ being trivially satisfied. A direct calculation from the commutation relations (3.18) of $\mathfrak{D}_{\mathrm{m|n}}$ leads to (for $i, j, l > 1$):

$$d_{j1}(u)d_{11}(v) \equiv 0 \mod \mathcal{I}_2, \tag{3.24}$$

$$[d_{ij}(u), d_{l1}(v)] \equiv 0 \mod \mathcal{I}_2, \tag{3.25}$$

$$[d_{ij}(u), d_{11}(v)] \equiv -\delta_{ij}\frac{\mathfrak{w}_{i1}(u, v)\bar{\mathfrak{w}}_{i1}(u, v)}{\mathfrak{b}(u, v)\bar{\mathfrak{b}}(u, v)}[d_{11}(u), d_{11}(v)] \mod \mathcal{I}_2, \tag{3.26}$$

$$[d_{11}(u), d_{11}(v)] \equiv 0 \mod \mathcal{I}_2. \tag{3.27}$$

Gathering all these equations, we obtain relation (3.20) for $k = 2$.

We now prove relation (3.23) for $k = 1, 2$. For $k = 1$, $d_{ij}^{(1)}(u) = d_{ij}(u)$, the ideal $\mathcal{I}_1$ is empty, and we have the starting algebra $\mathfrak{D}_{\mathrm{m|n}}$, so that relation (3.23) for $k = 1$ just corresponds to the standard reflection equation. For $k = 2$, we use the following commutation relations:

$$[d_{ij}(u), d_{11}(v)] \equiv 0 \quad \text{for} \quad i, j > 1 \quad \text{and} \quad [d_{11}(u), d_{11}(v)] \equiv 0, \tag{3.28}$$

$$d_{j1}(u) \equiv 0 \quad \text{for} \quad j > 1 \mod \mathcal{I}_2. \tag{3.29}$$

This implies that for $i, j, g, l > 1$, we have

$$\left[d_{ij}^{(2)}(u), d_{gl}^{(2)}(v)\right] \equiv [d_{ij}(u^{(1)}), d_{gl}(v^{(1)})] \mod \mathcal{I}_2. \tag{3.30}$$

Hence, it just remains to prove that the relation can be re-expressed in terms of $d_{rs}^{(1)}(u)$, $r, s > 1$, only. For such a purpose, we compute the commutation relations between $d_{i1}(u)$ and $d_{1j}(v)$:

$$d_{i1}(u)d_{1j}(v) \equiv (-1)^{[j]([j]+[i])}\frac{\mathfrak{w}_{1j}(u, v)}{\mathfrak{b}(u, v)}d_{11}(v)d_{ij}(u) + \delta_{ij}(-1)^{[j]+[1]}\frac{\bar{\mathfrak{w}}_{i1}(u, v)}{\bar{\mathfrak{b}}(u, v)}d_{11}(v)d_{11}(u)$$

$$- (-1)^{[1]+[j][i]}\frac{\mathfrak{w}_{i1}(u, v)}{\mathfrak{b}(u, v)}d_{11}(u)d_{ij}(v)$$

$$- \sum_{a=1}^{m+n}(-1)^{([i]+[j])[a]}\frac{\bar{\mathfrak{w}}_{1a}(u, v)}{\bar{\mathfrak{b}}(u, v)}d_{ia}(u)d_{aj}(v). \tag{3.31}$$

Using this equation, one can compute[2] for any polynomial function $f([a], [b], [c], [d])$:

$$\sum_{a=1}^{m+n}(-1)^{f([i],[l],[a],[g])}\frac{\bar{\mathfrak{w}}_{ga}(u^{(1)}, v^{(1)})}{\bar{\mathfrak{b}}(u^{(1)}, v^{(1)})}d_{ia}(u^{(1)})d_{al}(v^{(1)})$$

$$\equiv \sum_{a=2}^{m+n}(-1)^{f([i],[l],[a],[g])}\frac{\bar{\mathfrak{w}}_{ga}(u, v)}{\bar{\mathfrak{b}}(u, v)}d_{ia}(u^{(1)})d_{al}(v^{(1)})$$

$$- (-1)^{[1]+[g][i]+f([i],[l],[1],[g])}\frac{\mathfrak{w}_{i1}(u, v)\bar{\mathfrak{w}}_{g1}(u^{(1)}, v^{(1)})}{\mathfrak{b}(u, v)\bar{\mathfrak{a}}_1(u^{(1)}, v^{(1)})}d_{11}(u^{(1)})d_{il}(v^{(1)})$$

---

[2] To obtain this equation, we started from the l.h.s. of (3.32), and changed the term $a = 1$ according to (3.31).

$$+ (-1)^{[l]([l]+[i])+f([i],[l],[1],[g])} \frac{\mathfrak{w}_{1l}(u,v)\bar{\mathfrak{w}}_{g1}(u^{(1)},v^{(1)})}{\mathfrak{b}(u,v)\bar{\mathfrak{a}}_1(u^{(1)},v^{(1)})} d_{11}(v^{(1)})d_{il}(u^{(1)})$$

$$+ \delta_{il}(-1)^{[1]+[i]+f([i],[l],[1],[g])} \frac{\bar{\mathfrak{w}}_{i1}(u^{(1)},v^{(1)})\bar{\mathfrak{w}}_{g1}(u^{(1)},v^{(1)})}{\bar{\mathfrak{b}}(u^{(1)},v^{(1)})\bar{\mathfrak{a}}_1(u^{(1)},v^{(1)})} d_{11}(u^{(1)})d_{11}(v^{(1)}). \quad (3.32)$$

Now using (3.28), we can write the commutation relations for the new operator $d_{ij}^{(1)}(u)$ with (3.18). We extract the term $d_{i1}(u)d_{1j}(v)$ of each sum and using (3.32) we are left with only operators $d_{11}(u)$ and $d_{ij}(u)$ with $i, j \neq 1$. We use the transformation

$$d_{ij}^{(2)}(u) = d_{ij}^{(1)}(u^{(1)}) - \delta_{ij}\psi_1(u^{(1)})d_{11}^{(1)}(u^{(1)}), \quad (3.33)$$

and we obtain the desired commutation relation plus some unwanted terms. It is a straightforward calculation to find that the unwanted terms cancel for all values of $i, j, g, l \in \{2, \ldots, \mathfrak{m}+\mathfrak{n}\}$. This proves that $\hat{d}_{ij}(u)$ has the same commutation relation as (3.18) with a sum starting at 2 for $i, j > 1$. Thus, theorem 3.1 is proved for $k = 2$.

For $k > 2$, we first remark that one has $\mathcal{I}_k \subset \mathcal{I}_{k+1}$, so that one can use the results of step k in the proof of step $k + 1$. Then, the calculation becomes equivalent to the $k = 2$ case. The transformation of D operator for each step is

$$D^{(k)}(u) = \mathbb{I}^{(k)}\widehat{D}^{(k-1)}(u)\mathbb{I}^{(k)}, \qquad k > 1 \quad (3.34)$$

$$\widehat{D}^{(k)}(u) = \mathbb{I}^{(k)}\big(\widehat{D}^{(k-1)}(u^{(k-1)}) - \psi_{k-1}(u^{(k-1)})d_{kk}^{(k-1)}(u^{(k-1)})\mathbb{I}^{(k)}\big)\mathbb{I}^{(k)}, \qquad k > 1 \quad (3.35)$$

$$D^{(1)}(u) = \widehat{D}^{(1)}(u) = \mathbb{I}^{(1)}D(u)\mathbb{I}^{(1)} = D(u). \quad (3.36)$$

A direct calculation using identity (B.10) gives the form (3.22). $\qquad \square$

Lemma 3.2. *For $k = 1, 2, \ldots, \mathfrak{m}+\mathfrak{n}-1$, let $G^{(k)}$ be a linear combination of elements,*

$$t_{i_1 j_1}(u_1)t'_{g_1 l_1}(\iota(u_1)) \cdots t_{i_p j_p}(u_p)t'_{g_p l_p}(\iota(u_p)),$$

with all indices $i_r, j_r, g_r, l_r > k-1$, and let $\mathcal{J}_k$ be the left ideal generated by $\{t_{ij}(u), t'_{ij}(u)\}$ for $i > j$ and $j < k$. Then, we have the following properties:

$$t_{ij}(u)G^{(k)} \equiv 0 \mod \mathcal{J}_k \qquad for \quad i > j \quad and \quad j < k, \quad (3.37)$$

$$t'_{ij}(u)G^{(k)} \equiv 0 \mod \mathcal{J}_k \qquad for \quad i > j \quad and \quad j < k, \quad (3.38)$$

$$[t_{ii}(u), G^{(k)}] \equiv 0 \mod \mathcal{J}_k \qquad for \quad i < k, \quad (3.39)$$

$$[t'_{ii}(u), G^{(k)}] \equiv 0 \mod \mathcal{J}_k \qquad for \quad i < k. \quad (3.40)$$

Moreover, the generators,

$$T^{(k)}(u) = \sum_{i,j=k}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t_{ij}^{(k)}(u) \qquad and \qquad (T^{-1})^{(k)}(\iota(u)) = \sum_{i,j=k}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t_{ij}^{\prime(k)}(\iota(u)), \quad (3.41)$$

$$t_{ij}^{(k)}(u) = t_{ij}(u^{(1\ldots k-1)}) \qquad and \qquad t_{ij}^{\prime(k)}(\iota(u)) = t'_{ij}(\iota(u^{(1\ldots k-1)})), \quad (3.42)$$

satisfy in $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}/\mathcal{J}_k$ the relation:

$$\big(T_2^{-1}\big)^{(k)}(\iota(u))R_{12}^{(k)}(u, \iota(u))T_1^{(k)}(u) \equiv T_1^{(k)}(u)R_{12}^{(k)}(u, \iota(u))\big(T_2^{-1}\big)^{(k)}(\iota(u)) \mod \mathcal{J}_k. \quad (3.43)$$

Proof. As for theorem 3.1, the case $k = 1$ is just the definition of the algebra and relations (3.37)–(3.40) do not exist.

We prove the case $k = 2$, the proof for the other cases being similar. From the relation

$$T_2^{-1}(v) R_{12}(u, v) T_1(u) = T_1(u) R_{12}(u, v) T_2^{-1}(v) \tag{3.44}$$

one obtains by projecting on $E_{ij} \otimes E_{gl}$:

$$[t'_{ij}(u), t_{gl}(v)] = \delta_{il} \sum_{a=1}^{m+n} (-1)^{([j]+[a])([g]+[a])} \frac{\mathfrak{w}_{al}(v, u)}{\mathfrak{b}(v, u)} t_{ga}(v) t'_{aj}(u)$$

$$- \delta_{jg} \sum_{a=1}^{m+n} (-1)^{([g]+[a])([i]+[l])} \frac{\mathfrak{w}_{ga}(v, u)}{\mathfrak{b}(v, u)} t'_{ia}(u) t_{al}(v). \tag{3.45}$$

From (3.45) we find for $i, j, g, l \neq 1$:

$$t'_{i1}(u) t_{11}(v) \equiv 0 \mod \mathcal{J}_2; \quad [t'_{ij}(u), t_{g1}(v)] \equiv 0 \mod \mathcal{J}_2 \tag{3.46}$$

$$t_{g1}(v) t'_{11}(u) \equiv 0 \mod \mathcal{J}_2; \quad [t'_{i1}(u), t_{gl}(v)] \equiv 0 \mod \mathcal{J}_2. \tag{3.47}$$

We also need the following commutation relation proved in [16]:

$$[t_{ij}(u), t_{k1}(v)] \equiv 0 \mod \mathcal{J}_2. \tag{3.48}$$

In the same way, one can compute

$$[t'_{ij}(u), t'_{k1}(v)] \equiv 0 \mod \mathcal{J}_2. \tag{3.49}$$

Starting from the left-hand side of relations (3.37)–(3.40), a recursive use of commutation relations (3.46)–(3.49) proves that one gets only terms with $t_{a1}(u)$ and $t'_{a1}(u)$ on the right, so that properties (3.37)–(3.40) hold for $k = 2$.

To prove (3.43), we start again with relation (3.45) with $i, j, g, l \neq 1$ and extract the first term in the summation:

$$[t'_{ij}(u), t_{gl}(v)] = \delta_{il} \sum_{a=2}^{m+n} (-1)^{([j]+[a])([g]+[a])} \frac{\mathfrak{w}_{al}(v, u)}{\mathfrak{b}(v, u)} t_{ga}(v) t'_{aj}(u)$$

$$- \delta_{jg} \sum_{a=2}^{m+n} (-1)^{([g]+[a])([i]+[l])} \frac{\mathfrak{w}_{ga}(v, u)}{\mathfrak{b}(v, u)} t'_{ia}(u) t_{al}(v)$$

$$+ \delta_{il} (-1)^{[j][g]+[1]} \frac{\mathfrak{w}_{1l}(v, u)}{\mathfrak{b}(v, u)} t_{g1}(v) t'_{1j}(u)$$

$$- \delta_{jg} (-1)^{[g]([i]+[l])} \frac{\mathfrak{w}_{g1}(v, u)}{\mathfrak{b}(v, u)} t'_{i1}(u) t_{1l}(v). \tag{3.50}$$

Inserting in this equation the relations (valid modulo $\mathcal{J}_2$):

$$t'_{i1}(u) t_{1l}(v) \equiv - \sum_{a=2}^{m+n} (-1)^{[a]([i]+[l])} \frac{\mathfrak{w}_{1a}(v, u)}{\mathfrak{a}_1(v, u)} t'_{ia}(u) t_{al}(v) + \delta_{il} \frac{\mathfrak{w}_{1l}(v, u)}{\mathfrak{a}_1(v, u)} t'_{11}(u) t_{11}(v), \tag{3.51}$$

$$(-1)^{[1]+[i][j]} t_{i1}(v) t'_{1j}(u) \equiv - \sum_{a=2}^{m+n} (-1)^{([a]+[j])([a]+[i])} \frac{\mathfrak{w}_{a1}(v, u)}{\mathfrak{a}_1(v, u)} t_{ia}(v) t'_{aj}(u)$$

$$+ \delta_{ji} \frac{\mathfrak{w}_{1l}(v, u)}{\mathfrak{a}_1(v, u)} t'_{11}(u) t_{11}(v), \tag{3.52}$$

$$[t'_{11}(u), t_{11}(v)] \equiv 0, \tag{3.53}$$

and making the transformation $u \to \iota(u^{(1)})$ and $v \to u^{(1)}$ it is straightforward to end the proof. For $k > 2$ we use the same argument as in the proof of theorem 3.1. $\qquad \square$

10

Theorem 3.3. *In the coset* $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}/\mathcal{J}_k \otimes \mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}/\mathcal{I}_k$, *the coproduct takes the form*

$$\Delta\big(D_{[1]}^{(k)}(u)\big) \equiv T_{[2]}^{(k)}(u) D_{[1]}^{(k)}(u) \big(T_{[2]}^{-1}\big)^{(k)}(\iota(u)) \mod \mathcal{J}_k, \tag{3.54}$$

where [1] *labels the space* $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}/\mathcal{I}_k$, [2] *labels the space* $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}/\mathcal{J}_k$ *and* Δ *is the coproduct of* $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$.

Proof. As in theorem 3.1, we just do the proof for $k = 2$, the other cases follow. From

$$\Delta(d_{ij}(u^{(1)}) - \delta_{ij}\psi_1(u^{(1)})d_{11}(u^{(1)})) = \sum_{a,b=1}^{\mathfrak{m}+\mathfrak{n}} \{t_{ia}(u^{(1)})t'_{bj}(\iota(u^{(1)})) \otimes d_{ab}(u^{(1)})$$

$$- \delta_{ij}\psi_1(u^{(1)})t_{1a}(u^{(1)})t'_{b1}(\iota(u^{(1)})) \otimes d_{ab}(u^{(1)})\},$$

and using the quotient $\mathcal{I}_2$ and $\mathcal{J}_2$, it follows:

$$\Delta(d_{ij}(u^{(1)}) - \delta_{ij}\psi_1(u^{(1)})d_{11}(u^{(1)})) = \sum_{a,b=2}^{\mathfrak{m}+\mathfrak{n}} t_{ia}(u^{(1)})t'_{bj}(\iota(u^{(1)})) \otimes d_{ab}(u^{(1)})$$

$$+ t_{i1}(u^{(1)})t'_{1j}(\iota(u^{(1)})) \otimes d_{11}(u^{(1)}) - \delta_{ij}\psi_1(u^{(1)})t_{11}(u^{(1)})t'_{11}(\iota(u^{(1)})) \otimes d_{11}(u^{(1)}).$$

Using the commutation relation (3.45) for the second term we find

$$\Delta(d_{ij}(u^{(1)}) - \delta_{ij}\psi_1(u^{(1)})d_{11}(u^{(1)}))$$

$$= \sum_{a,b=2}^{\mathfrak{m}+\mathfrak{n}} t_{ia}(u^{(1)})t'_{bj}(\iota(u^{(1)})) \otimes (d_{ij}(u^{(1)}) - \delta_{ij}\psi_1(u^{(1)})d_{11}(u^{(1)})).$$

Theorem 3.1 and lemma 3.2 allow us to generalize this result to each k.　　□

4. Highest weight representations

The fundamental point in using the ABA is to know a pseudo-vacuum for the model. In the mathematical framework it is equivalent to knowing a highest weight representation for the algebra which underlies the model. Since the generators of the algebra $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ can be constructed from the $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ ones, see equation (3.10), we first describe how to construct highest representations for the infinite-dimensional (graded) algebras $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ from the highest weight representation of the finite-dimensional Lie subalgebras $gl(\mathfrak{m}|\mathfrak{n})$ or $\mathcal{U}_q(\mathfrak{m}|\mathfrak{n})$. Next, we show how these representations induce (for diagonal $K^-(u)$ matrix) a representation for $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ with same highest weight vector.

4.1. Finite-dimensional representations of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$

Definition 4.1. *A representation of* $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ *is called the highest weight if there exists a nonzero vector* Ω *such that*

$$t_{ii}(u)\Omega = \lambda_i(u)\Omega \qquad and \qquad t_{ij}(u)\Omega = 0 \qquad for \quad i > j, \tag{4.1}$$

for some scalars $\lambda_i(u) \in \mathbb{C}[[u^{-1}]]$. $\lambda(u) = (\lambda_1(u), \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}}(u))$ *is called the highest weight and* Ω *the highest weight vector.*

It is known (see [27–30]) that any finite-dimensional irreducible representation of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ is the highest weight and that it contains a unique (up to scalar multiples) highest weight vector. To construct such representations, one uses the evaluation morphism, which relates the infinite-dimensional algebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ to its finite-dimensional subalgebra $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ (see [16]). From the

evaluation morphism ev_a (with $a \in \mathbb{C}$) and a highest weight representation π_μ of $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ (where μ is a $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ highest weight), one can construct a highest weight representation of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, called the evaluation representation:

$$\rho_a^\mu = ev_a \circ \pi_\mu : \mathcal{A}_{\mathfrak{m}|\mathfrak{n}} \xrightarrow{ev_a} \mathcal{B}_{\mathfrak{m}|\mathfrak{n}} \xrightarrow{\pi_\mu} \mathcal{V}_\lambda. \tag{4.2}$$

The weight of this evaluation representation is given by $\lambda(u) = (\lambda_1(u), \ldots, \lambda_{\mathfrak{m}+\mathfrak{n}}(u))$, with

$$\lambda_j(u) = \begin{cases} u - a - (-1)^{[j]}\hbar\mu_j & \text{for} \quad \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ (-1)^{[j]}\left(\dfrac{u}{a}\eta_j q^{\mu_j} - \dfrac{a}{u}\eta_j q^{-\mu_j}\right) & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}) \end{cases} \quad j = 1, \ldots, \mathfrak{m}+\mathfrak{n}, \tag{4.3}$$

where μ_j, $j = 1, \ldots, \mathfrak{m}+\mathfrak{n}$ are the weights of the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ representation. More generally, one constructs the tensor product of evaluation representations using the coproduct of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$,

$$\left(\otimes_{i=1}^{L}\rho_{a_i}^{\mu^{\langle i\rangle}}\right) \circ \Delta^{(L)}(T(u)) = \rho_{a_1}^{\mu^{\langle 1\rangle}}(T(u)) \dot{\otimes} \rho_{a_2}^{\mu^{\langle 2\rangle}}(T(u)) \dot{\otimes} \cdots \dot{\otimes} \rho_{a_L}^{\mu^{\langle L\rangle}}(T(u)), \tag{4.4}$$

where $\mu^{\langle i\rangle} = \left(\mu_1^{\langle i\rangle}, \ldots, \mu_{\mathfrak{m}+\mathfrak{n}}^{\langle i\rangle}\right)$, $i = 1, \ldots, L$, are the weights of the $\mathcal{B}_{\mathfrak{m}|\mathfrak{n}}$ representations. This provides a $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ representation with weight,

$$\lambda_j(u) = \prod_{i=1}^{L} \lambda_j^{\langle i\rangle}(u), \qquad j = 1, \ldots, \mathfrak{m}+\mathfrak{n}, \tag{4.5}$$

where $\lambda_j^{\langle i\rangle}(u)$ have the form (4.3).

4.2. Representations of $T^{-1}(u)$ from $T(u)$

The construction of the finite-dimensional representations for $T^{-1}(u)$ in relation with the $T(u)$ ones is different for the $gl(\mathfrak{n})$ and the super symmetric cases $gl(\mathfrak{m}|\mathfrak{n})$. For the $gl(\mathfrak{n})$ algebra, the representations are constructed using the quantum determinant $q\det(T(u))$ and the comatrix $\widehat{T}(u)$ see [31], while for the $gl(\mathfrak{m}|\mathfrak{n})$ superalgebra, one uses the Liouville contraction, the quantum Berezinian $Ber(T(u))$ [32] and the crossing symmetry of $T(u)$.

We define for this section:

$$u_{\{k\}} = \begin{cases} u + \hbar k \\ uq^{-k} \end{cases} \quad \text{and} \quad f_{ij}(\sigma) = \begin{cases} (-1)^{i+j+1}s(\sigma) & \text{for} \quad \mathcal{Y}(\mathfrak{n}) \text{ and } \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ (-q)^{l(\sigma)+i-j} & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{n}) \text{ and } \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}), \end{cases} \tag{4.6}$$

where $s(\sigma)$ is the sign of the permutation σ and $l(\sigma)$ its length.

$\mathcal{A}_\mathfrak{n}$ *case.* We use the $\mathcal{A}_\mathfrak{n}$ quantum determinant $q\det(T(u))$ which generates the centre of $\mathcal{A}_\mathfrak{n}$,

$$q\det(T(u)) = \sum_{\sigma \in S_\mathfrak{n}} f_{00}(\sigma) \prod_{i=1}^{\mathfrak{n}} t_{i\sigma(i)}(u_{\{i-\mathfrak{n}\}}), \tag{4.7}$$

and the quantum comatrix,

$$\widehat{T}(u) = \sum_{ij=1} E_{ij} \otimes \widehat{t}_{ij}(u) \tag{4.8}$$

$$\widehat{t}_{ij}(u) = \sum_{\sigma \in S_\mathfrak{n}} f_{ij}(\sigma)^{-1} t_{1a_{\sigma(1)}}(u_{\{2-\mathfrak{n}\}}) \cdots t_{i-1a_{\sigma(i-1)}}(u_{\{i-\mathfrak{n}\}}) t_{i+1a_{\sigma(i)}}(u_{\{i+1-\mathfrak{n}\}}) \cdots t_{\mathfrak{n}a_{\sigma(\mathfrak{n}-1)}}(u)$$

$$\text{with} \quad (a_1, \ldots, a_{\mathfrak{n}-1}) = (1, \ldots, j-1, j+1, \ldots, \mathfrak{n}) \tag{4.9}$$

which obeys $\widehat{T}(u)T(u_{\{1-\mathfrak{n}\}}) = q\det(T(u))$. This equation allows to relate $T^{-1}(u)$ to $\widehat{T}(u)$:

$$T^{-1}(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} E_{ij} \otimes t'_{ij}(u) = \frac{\widehat{T}(u_{\{\mathfrak{n}-1\}})}{q\det(T(u_{\{\mathfrak{n}-1\}}))}. \tag{4.10}$$

To write the form of the highest weight irreducible representation for $T^{-1}(u)$, one first computes the action of $q \det(T(u))$ and $\widehat{t}_{ii}(u)$ on Ω:

$$q \det(T(u))\Omega = \prod_{i=1}^{\mathfrak{n}} \lambda_i(u_{\{i-\mathfrak{n}\}})\Omega, \tag{4.11}$$

$$\widehat{t}_{ii}(u)\Omega = \lambda_1(u_{\{2-\mathfrak{n}\}})\cdots\lambda_{i-1}(u_{\{i-\mathfrak{n}\}})\lambda_{i+1}(u_{\{i+1-\mathfrak{n}\}})\cdots\lambda_{\mathfrak{n}}(u)\Omega. \tag{4.12}$$

Then, since $\widehat{t}_{ij}(u)\Omega = 0$ for $i > j$, one finds

$$t'_{ii}(u)\Omega = \lambda'_i(u)\Omega \qquad \text{with} \quad \lambda'_i(u) = \left(\prod_{k=1}^{i-1} \frac{\lambda_k(u_{\{k\}})}{\lambda_k(u_{\{k-1\}})}\right) \frac{1}{\lambda_i(u_{\{i-1\}})}, \tag{4.13}$$

$$t'_{ij}(u)\Omega = 0 \quad \text{if} \quad i > j.$$

$\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ *case.* First, one has to prove that Ω is a highest weight vector of $T^{-1}(u)$. The proof is done in [15] for the super-Yangian case. The quantum superalgebra $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$ case is done in the following theorem:

Theorem 4.2. *For the quantum superalgebra $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$, the highest weight vector Ω of $T(u)$ is also a highest weight vector of $T^{-1}(u)$.*

$$t'_{ii}(u)\Omega = \lambda'_i(u)\Omega \qquad and \qquad t'_{ij}(u)\Omega = 0 \quad if \quad i > j. \tag{4.14}$$

Proof. To prove this theorem we must use the commutation relation between the modes of $T(u) = L^+(u)$ and $T^{-1}(u) = (L^+)^{-1}(u) = \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} \bar{L}_{ij}^+(u)$. As $L^+(u)$ is a formal Taylor series in u, its inverse is also a formal Taylor series of u:

$$L_{ij}^+(u) = \sum_{n=0}^{\infty} L_{ij}^{(n)} u^{2n} \qquad \text{and} \qquad \bar{L}_{ij}^+(u) = \sum_{n=0}^{\infty} \bar{L}_{ij}^{(n)} u^{2n}. \tag{4.15}$$

Projecting the commutation relation (3.45) on the modes we find the following relation:

$$\left[\bar{L}_{ij}^{(p)}; L_{kl}^{(q)}\right\} = \delta_{il} \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{([a]+[j])([a]+[k])} \left(c_{al}^+ \sum_{b=0}^{p} L_{ka}^{(q+b)} \bar{L}_{aj}^{(p-b)} - c_{al}^- \sum_{b=1}^{p} L_{ka}^{(q+b)} \bar{L}_{aj}^{(p-b)}\right)$$

$$- \delta_{jk} \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{([i]+[l])([a]+[k])} \left(c_{ka}^+ \sum_{b=0}^{p} \bar{L}_{ia}^{(p-b)} L_{al}^{(q+b)} - c_{ka}^- \sum_{b=1}^{p} \bar{L}_{ia}^{(p-b)} L_{al}^{(q+b)}\right),$$

$$\text{with} \qquad c_{al}^\pm = q^{\pm(1-2[l])} - q^{\text{sign}(l-a)(1-2[l])}. \tag{4.16}$$

From the relation $T(u)T^{-1}(u) = T^{-1}(u)T(u) = \mathbb{I}$ we find

$$\sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \sum_{q=0}^{p} (-1)^{([i]+[a])([a]+[j])} \bar{L}_{ia}^{(q)} L_{aj}^{(p-q)} = \sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} \sum_{q=0}^{p} (-1)^{([i]+[a])([a]+[j])} L_{ia}^{(q)} \bar{L}_{aj}^{(p-q)} = \delta_{ij}\delta_{0p}. \tag{4.17}$$

We also know that $L_{ij}^{(0)} = 0$ for $i < j$ and that $L_{ii}^{(0)}$ has an inverse. First we prove the same properties for $\bar{L}^{(0)}$. From (4.17), we deduce

$$\sum_{a=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{([i]+[a])([a]+[j])} L_{ia}^{(0)} \bar{L}_{aj}^{(0)} = 0 \qquad \text{for} \quad i \neq j. \tag{4.18}$$

13

As $L_{ii}^{(0)} \neq 0$, we find for $i = 1$, $\bar{L}_{1j}^{(0)} = 0$. By induction on i, we find $\bar{L}_{ij}^{(0)} = 0$ for $i < j$. Then,

$$\sum_{a=1}^{m+n}(-1)^{[i]+[a]}L_{ia}^{(0)}\bar{L}_{ai}^{(0)} = \sum_{a=1}^{m+n}(-1)^{[i]+[a]}\bar{L}_{ia}^{(0)}L_{ai}^{(0)} = 1 \qquad (4.19)$$

implies that $L_{ii}^{(0)}\bar{L}_{ii}^{(0)} = \bar{L}_{ii}^{(0)}L_{ii}^{(0)} = 1$.

Now, we have to prove that Ω is a highest weight vector of $\bar{L}_{ij}^{(p)}$. We already know that

$$L_{ij}^{(p)}\Omega = 0 \qquad \text{for} \quad i < j \qquad \text{and} \qquad L_{ii}^{(p)}\Omega = \lambda_i^{(p)}\Omega. \qquad (4.20)$$

We can write from (4.17) with $i > j$:

$$\sum_{q=0}^{p}\lambda_j^{(p-q)}\bar{L}_{ij}^{(q)}\Omega = -\sum_{a=1}^{j-1}\sum_{q=0}^{p-1}(-1)^{([k]+[a])([i]+[j])}\bar{L}_{ia}^{(q)}L_{aj}^{(p-q)}\Omega. \qquad (4.21)$$

To prove that $L_{ij}^{(p)}\Omega = 0$ for $i < j$, we use a double induction, on p and on i. We already proved directly that $\bar{L}_{ij}^{(0)}\Omega = 0$, $i < j$. For $p = 1$ we have

$$\lambda_j^{(0)}\bar{L}_{ij}^{(1)}\Omega = -\sum_{a=1}^{j-1}(-1)^{([k]+[a])([i]+[j])}\bar{L}_{ia}^{(0)}L_{aj}^{(1)}\Omega = -\sum_{a=1}^{j-1}(-1)^{([k]+[a])([i]+[j])}\big[\bar{L}_{ia}^{(0)}, L_{aj}^{(1)}\big\}\Omega.$$

Using the commutation relations (4.16) we find

$$\big[\bar{L}_{ia}^{(0)}, L_{aj}^{(1)}\big\}\Omega = -q^{-1+2[a]}(q-q^{-1})(-1)^{([i]+[j])[a]}\sum_{b=1}^{a-1}(-1)^{[b]([i]+[b]+[j])}\big[\bar{L}_{ib}^{(0)}, L_{bj}^{(1)}\big\}\Omega. \quad (4.22)$$

We get a triangular system in a, so that the property is proved for $p = 1$ by induction on a. For a general p we use the same method.

Finally, we prove that $\bar{L}_{ii}^{(p)}\Omega = \bar{\lambda}_i^{(p)}\Omega$. For $p = 0$, from the equation $\bar{L}_{ii}^{(0)}L_{ii}^{(0)} = 1$, we already know that $\bar{L}_{ii}^{(0)}\Omega = \big(\lambda_i^{(0)}\big)^{-1}\Omega$. We prove the property of the general case by a double induction, assuming the property is true for $p - 1$ and starting from

$$\sum_{q=0}^{p}\lambda_i^{(p-q)}\bar{L}_{ii}^{(q)}\Omega = \left(\delta_{0,p} - \sum_{a=1}^{i-1}\sum_{q=0}^{p-1}(-1)^{[i]+[a]}\big[\bar{L}_{ia}^{(q)}, L_{ai}^{(p-q)}\big\}\right)\Omega. \qquad (4.23)$$

From the commutation relation we obtain

$$\big[\bar{L}_{ia}^{(q)}, L_{ai}^{(p-q)}\big\}\Omega = \sum_{b=1}^{a}\left(c_{bi}^{-}\sum_{r=1}^{q}\big[\bar{L}_{ib}^{(q-r)}, L_{bi}^{(p-q+r)}\big\} - c_{ab}^{+}\sum_{r=0}^{q}\big[\bar{L}_{ib}^{(q-r)}, L_{bi}^{(p-q+r)}\big\}\right)\Omega,$$

$$\big[\bar{L}_{ia}^{(0)}, L_{ai}^{(q)}\big\}\Omega = -q^{-1+2[a]}(q-q^{-1})(-1)^{[a]}\sum_{b=1}^{a-1}\big[\bar{L}_{ib}^{(0)}, L_{bi}^{(q)}\big\}\Omega = 0. \qquad (4.24)$$

From the second equation, equal to zero by induction on a, we find

$$\big[\bar{L}_{ia}^{(q)}, L_{ai}^{(p-q)}\big\}\Omega = -q^{-1+2[a]}(q-q^{-1})(-1)^{[a]}\sum_{b=1}^{a-1}\sum_{r=1}^{q}\big[\bar{L}_{ib}^{(q-r)}, L_{bi}^{(p-q+r)}\big\}\Omega. \qquad (4.25)$$

By induction on q then on a we find the last equality equals zero. Thus, we have the relation:

$$\sum_{q=0}^{p}\lambda_i^{(p-q)}\bar{L}_{ii}^{(q)}\Omega = \delta_{0,p}\Omega. \qquad (4.26)$$

It follows that $\bar{L}_{ii}^{(q)}\Omega = \bar{\lambda}_i^{(q)}\Omega$ with $\bar{\lambda}_i^{(q)}$ rational function of $\lambda_i^{(q)}, \ldots, \lambda_i^{(0)}$. $\qquad\qquad\square$

14

Second, we use the crossing symmetry of the monodromy matrix $T_a(u)$ and the quantum Berezinian to give an explicit expression of the weight of $T^{-1}(u)$. Let us introduce

$$T^*(u) = (T^{-1}(u))^t. \tag{4.27}$$

The crossing symmetry takes the form for $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ (see [15, 33]):

$$(T^t(u))^{-1} = \frac{1}{Z(u_{\{\mathfrak{n}-\mathfrak{m}\}})} M U T^*(u_{\{\mathfrak{n}-\mathfrak{m}\}}) U M^{-1} \qquad \text{with} \quad U = \sum_{i=1}^{\mathfrak{m}+\mathfrak{n}} (-1)^{[i]} E_{ii}, \tag{4.28}$$

where the Liouville contraction $Z(u)$ lies in the centre of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$. It can be written in terms of the Berezinian, that itself relies on the quantum determinant (4.7):

$$\text{Ber}(T(u)) = q \det(T^{(\mathfrak{m})}(u_{\{\mathfrak{m}-\mathfrak{n}-1\}})) q \, \widetilde{\det}((T^*)^{(\mathfrak{n})}(u_{\{-\mathfrak{n}\}}))$$

$$= \prod_{i=1}^{\mathfrak{m}} \lambda_i(u_{\{i-\mathfrak{n}-1\}}) \prod_{i=1}^{\mathfrak{n}} \lambda'_{i+\mathfrak{m}}(u_{\{i-\mathfrak{n}-1\}}) \tag{4.29}$$

$$Z(u) = \frac{\text{Ber}(T(u_{\{1\}}))}{\text{Ber}(T(u))} \qquad \text{with} \quad q \, \widetilde{\det}(T^{(\mathfrak{n})}(u))$$

$$= \sum_{\sigma \in S_{\mathfrak{n}}} f_{00}(\sigma)^{-1} \prod_{i=1}^{\mathfrak{n}} t_{i+\mathfrak{m},\sigma(i)+\mathfrak{m}}(u_{\{\mathfrak{n}-i\}}), \tag{4.30}$$

with $T^{(k)}(u) = \mathbb{I}^{(k)} T(u) \mathbb{I}^{(k)}, \forall k$ and $\mathbb{I}^{(k)}$ defined in (2.14). In $\mathcal{A}_{0|\mathfrak{n}} \subset \mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, we have

$$(T^{(\mathfrak{n})}(u)^t)^{-1} = z(u) M T^{(\mathfrak{n})*}(u_{\{\mathfrak{n}\}}) M^{-1}, \tag{4.31}$$

where $z(u)$, the $\mathcal{A}_{0|\mathfrak{n}}$ Liouville contraction, can be written in terms of the quantum determinant:

$$z(u) = \frac{q \, \widetilde{\det}(T^{(\mathfrak{n})}(u_{\{1\}}))}{q \, \widetilde{\det}(T^{(\mathfrak{n})}(u))} = \prod_{i=1}^{\mathfrak{n}} \frac{\lambda_{i+\mathfrak{m}}(u_{\{\mathfrak{n}-i+1\}})}{\lambda_{i+\mathfrak{m}}(u_{\{\mathfrak{n}-i\}})}. \tag{4.32}$$

Lemma 4.3. *For the superalgebra $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, we have*

$$t'_{ii}(u)\Omega = \lambda'_i(u)\Omega \qquad and \qquad t'_{ij}(u)\Omega = 0 \qquad for \quad i > j,$$

$$with \qquad \lambda'_i(u) = \begin{cases} \dfrac{1}{\lambda_i(u_{\{i-1\}})} \left(\displaystyle\prod_{k=1}^{i-1} \dfrac{\lambda_k(u_{\{k\}})}{\lambda_k(u_{\{k-1\}})} \right) & for \quad 1 \leqslant i \leqslant \mathfrak{m} \\[20pt] \dfrac{Z(u)}{\lambda_i(u_{\{2\mathfrak{m}-i\}})} \left(\displaystyle\prod_{k=i+1}^{\mathfrak{m}+\mathfrak{n}} \dfrac{\lambda_k(u_{\{2\mathfrak{m}+1-k\}})}{\lambda_k(u_{\{2\mathfrak{m}-k\}})} \right) & for \quad \mathfrak{m}+1 \leqslant i \leqslant \mathfrak{m}+\mathfrak{n}. \end{cases} \tag{4.33}$$

Proof. From theorem 4.2, we have

$$T^{-1}(u)\Omega = \begin{pmatrix} T^{-1}(u)^{(\mathfrak{m})} & * \\ 0 & T^{-1}(u)^{(\mathfrak{n})} \end{pmatrix} \Omega, \tag{4.34}$$

$$T^*(u)\Omega = \begin{pmatrix} T^*(u)^{(\mathfrak{m})} & 0 \\ * & T^*(u)^{(\mathfrak{n})} \end{pmatrix} \Omega. \tag{4.35}$$

Multiplying (4.34) by $T(u)$ and (4.35) by $U M^{-1} T^t(u_{\{\mathfrak{n}-\mathfrak{m}\}}) M U$, one obtains

$$T^{(\mathfrak{m})}(u)(T^{-1})^{(\mathfrak{m})}(u)\Omega = \Omega$$

and

$$(M^{-1})^{(\mathfrak{n})}(T^t)^{(\mathfrak{n})}(u_{\{\mathfrak{n}-\mathfrak{m}\}}) M^{(\mathfrak{n})}(T^*)^{(\mathfrak{n})}(u)\Omega = Z(u)\Omega.$$

Finally, upon multiplication by $T^{(\mathfrak{m})}(u)^{-1}$ and $(T^t)^{(\mathfrak{n})}(u)^{-1}$, one is led to

$$(T^{-1})^{(\mathfrak{m})}(u)\Omega = (T^{(\mathfrak{m})})^{-1}(u)\Omega$$

and

$$(T^*)^{(\mathfrak{n})}(u)\Omega = Z(u)z(u_{\{\mathfrak{m}-\mathfrak{n}\}})(T^{(\mathfrak{n})})^*(u_{\{\mathfrak{m}\}})\Omega \qquad (4.36)$$

that gives the lemma. $\square$

4.3. Finite-dimensional representations of $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ from $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$ ones

For the study of the representations of the reflection algebra, we follow essentially the lines given in [26] for the reflection algebra based on the Yangian of $gl(\mathfrak{n})$ and in [15] for the reflection algebra based on the super-Yangian of $gl(\mathfrak{m}|\mathfrak{n})$.

Theorem 4.4. *If Ω is a highest weight vector of $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}}$, with eigenvalue $(\lambda_1(u), \dots, \lambda_{\mathfrak{m}+\mathfrak{n}}(u))$, then, when $K^-(z) = \mathrm{diag}\big(\kappa_1(u), \dots, \kappa_{\mathfrak{m}+\mathfrak{n}}(u)\big)$, Ω is also a highest weight vector for $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$,*

$$d_{ij}(u)\Omega = 0 \qquad \text{for} \quad i > j, \qquad \text{and} \qquad d_{ii}(u)\Omega = \Lambda_i(u)\Omega, \qquad (4.37)$$

with eigenvalues:

$$\Lambda_i(u) = \mathcal{K}_i(u)\lambda_i(u)\lambda_i'(\iota(u)) + \sum_{k=1}^{i-1} \psi_k(u^{\overline{(1\dots k-1)}})\mathcal{K}_k(u)\lambda_k(u)\lambda_k'(\iota(u)), \qquad (4.38)$$

$$\mathcal{K}_i(u) = \kappa_i(u) - \sum_{k=1}^{i-1} \kappa_k(u) \frac{\mathfrak{w}_{ik}(u^{\overline{(1\dots k-1)}}, \iota(u^{\overline{(1\dots k-1)}}))}{\mathfrak{a}_{i-1}(u^{\overline{(1\dots i-2)}}, \iota(u^{\overline{(1\dots i-2)}}))} q^{i-k-1-2\sum_{l=k+1}^{i-1}[l]}. \qquad (4.39)$$

Proof. First, we prove $d_{ij}(u)\Omega = 0$ for $j < i$. One computes

$$d_{ij}(u)\Omega = \sum_{a=1}^{j-1} (-1)^{([i]+[a])([a]+[j])}\kappa_a(u)t_{ia}(u)t_{aj}'(\iota(u))\Omega \qquad (4.40)$$

$$= -\sum_{a=1}^{j-1} \kappa_a(u)[t_{aj}'(\iota(u)), t_{ia}(u)\}\Omega.$$

Applying the super-commutator on Ω with the constraint $a \leqslant j < i$, one obtains

$$[t_{aj}'(u), t_{ia}(\iota(u))\}\Omega = -\sum_{b=1}^{j-1} \frac{\mathfrak{w}_{ba}(u, \iota(u))}{\mathfrak{b}(u, \iota(u))}[t_{bj}'(\iota(u)), t_{ib}(u)\}\Omega. \qquad (4.41)$$

Considering the case $a = j$, one obtains

$$\sum_{b=1}^{j-1}[t_{bj}'(\iota(u)), t_{ib}(u)\}\Omega = 0. \qquad (4.42)$$

Plugging this result in the former equation, we obtain

$$[t_{aj}'(u), t_{ia}(\iota(u))\}\Omega = \frac{\mathfrak{w}_{a-1,a}(u, \iota(u)) - \mathfrak{w}_{a+1,a}(u, \iota(u))}{\mathfrak{a}_a(u, \iota(u)) - \mathfrak{w}_{a-1,a}(u, \iota(u))} \sum_{b=a+1}^{j-1}[t_{bj}'(\iota(u)), t_{ib}(u)\}\Omega. \qquad (4.43)$$

By iteration $(a = j-1, \dots, a = 1)$ one finds

$$[t_{aj}'(u), t_{ia}(\iota(u))\}\Omega = 0, \qquad (4.44)$$

which proves that $d_{ij}(u)\omega = 0$, $j < i$.

Second, we prove $d_{ii}(u)\Omega = \Lambda_i(u)\Omega$. Acting on Ω with $d_{ii}(u)$ one obtains

$$d_{ii}(u)\Omega = \kappa_i(u)\lambda_i(u)\lambda_i'(\iota(u))\Omega + \sum_{a=1}^{i-1}(-1)^{[i]+[a]}\kappa_a(u)t_{ia}(u)t_{ai}'(\iota(u))\Omega. \tag{4.45}$$

One can restrict this problem to the computation of $t_{ia}(u)t_{ai}'(\iota(u))\Omega$ for $i > a$ in terms of the eigenvalues $\lambda_i(u)\lambda_i'(\iota(u))$. From the relation (3.45), we obtain

$$(-1)^{[i]+[a]}t_{ia}(u)t_{ai}'(\iota(u))\Omega = -\sum_{b=1}^{i}(-1)^{[i]+[b]}\frac{\mathfrak{w}_{ba}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}t_{ib}(u)t_{bi}'(\iota(u))\Omega$$

$$+\sum_{b=1}^{a}\frac{\mathfrak{w}_{ib}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}t_{ab}'(\iota(u))t_{ba}(u)\Omega. \tag{4.46}$$

Applying (3.45) on Ω for $i = j = k = l$, one finds the identity:

$$\sum_{b=1}^{i-1}\frac{\mathfrak{w}_{ib}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}t_{ib}'(\iota(u))t_{bi}(u)\Omega = \sum_{b=1}^{i-1}(-1)^{[i]+[b]}\frac{\mathfrak{w}_{bi}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}t_{ib}(u)t_{bi}'(\iota(u))\Omega.$$

Let $F_{ij} = t_{ij}(u)t_{ji}'(\iota(u))\Omega$. Using the two previous equations, one finds for $j < i$:

$$F_{ij} = (-1)^{[i]+[j]}\frac{\mathfrak{w}_{ij}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}(F_{jj} - F_{ii}) + \sum_{a=1}^{j-1}(-1)^{[i]+[a]}\frac{\mathfrak{w}_{ai}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}F_{ja}$$

$$-\sum_{a=1}^{i-1}(-1)^{[j]+[a]}\frac{\mathfrak{w}_{aj}(u,\iota(u))}{\mathfrak{b}(u,\iota(u))}F_{ia}. \tag{4.47}$$

It is then easy (but lengthy) to show that the solution is

$$F_{ij} = (-1)^{[i]+[j]}\mathfrak{m}_{ij}(u^{\overline{(1...j-1)}},\iota(u^{\overline{(1...j-1)}}))\left[\frac{F_{jj}}{\mathfrak{a}_j(u^{\overline{(1...j-1)}},\iota(u^{\overline{(1...j-1)}}))}\right.$$

$$-\frac{q^{(i-j-1-2\sum_{a=j+1}^{i-1}[a])}}{\mathfrak{a}_{i-1}(u^{\overline{(1...i-2)}},\iota(u^{\overline{(1...i-2)}}))}F_{ii}$$

$$-\sum_{a=j+1}^{i-1}\frac{\mathfrak{m}_{ia}(u^{\overline{(1...a-1)}},\iota(u^{\overline{(1...a-1)}}))q^{(a-j-1-2\sum_{l=j+1}^{a-1}[l])}}{\mathfrak{a}_j(u^{\overline{(1...a-1)}},\iota(u^{\overline{(1...a-1)}}))\mathfrak{a}_{a-1}(u^{\overline{(1...a-2)}},\iota(u^{\overline{(1...a-2)}}))}F_{aa}\left.\right]. \tag{4.48}$$

One must use relations (B.7)–(B.9) between functions. Plugging the value of F_{ik} into equation (4.45), after some rearrangement one gets the eigenvalues $\Lambda_i(u)$. $\qquad\square$

5. Algebraic Bethe ansatz for $\mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$ with $\mathfrak{m} + \mathfrak{n} = 2$

In this section, we recall the framework of the algebraic Bethe ansatz (ABA) [34] introduced in order to compute transfer matrix eigenvalues and eigenvectors. For $\mathfrak{m} + \mathfrak{n} = 2$, one can consider three different algebras: $\mathfrak{D}_{0|2}$, $\mathfrak{D}_{2|0}$ and $\mathfrak{D}_{1|1}$. The method follows the same steps as the closed chain case, up to a preliminary step. We write the monodromy matrix in the following matricial form:

$$D(u) = \begin{pmatrix} d_{11}(u) & d_{12}(u) \\ d_{21}(u) & d_{22}(u) \end{pmatrix}. \tag{5.1}$$

In the open case the transfer matrix have the form:

$$d(u) = \mathrm{str}\big(K_a^+(u)D_a(u)\big) = (-1)^{[1]}m_1 k(u)d_{11}(u) + (-1)^{[2]}m_2 d_{22}(u), \quad (5.2)$$

$$K^+(u) = MK(u). \quad (5.3)$$

The matrix K is constructed from the solution (3.9):

$$K(u) = \begin{cases} \mathbb{I} & \text{for } a_+ = 0 \\ \mathrm{diag}(k(u), 1) & \text{for } a_+ = 1 \end{cases} \quad \text{with} \quad k(u) = \begin{cases} \dfrac{-u - \frac{\mathfrak{m}-\mathfrak{n}}{2}\hbar - c_+}{u + \frac{\mathfrak{m}-\mathfrak{n}}{2}\hbar - c_+} & \text{for } \mathcal{Y}(\mathfrak{m}|\mathfrak{n}) \\ \dfrac{u^{-2}q^{-(\mathfrak{m}-\mathfrak{n})} - c_+^2}{u^2 q^{\mathfrak{m}-\mathfrak{n}} - c_+^2} & \text{for } \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}). \end{cases} \quad (5.4)$$

Remark that for the particular case $\mathfrak{m} + \mathfrak{n} = 2$, the form chosen for $K^+(u)$ exhausts all possible diagonal solutions. We recall that for $K^-(u)$ we keep the general diagonal solution (3.9). Let Ω be the pseudo-vacuum state:

$$d_{11}(u)\Omega = \Lambda_1(u)\Omega, \qquad d_{22}(u)\Omega = \Lambda_2(u)\Omega, \qquad d_{21}(u)\Omega = 0. \quad (5.5)$$

Looking at the commutation relations (3.18) for $\mathfrak{m} + \mathfrak{n} = 2$, one can see that the $d_{22}(u)d_{12}(v)$ exchange relation is not symmetric to the $d_{11}(u)d_{12}(v)$ one. In order to compensate this asymmetry, we perform a change of basis and a shift,

$$d_{11}(u^{(1)}) = \widehat{d}_{11}(u), \qquad d_{12}(u^{(1)}) = \widehat{d}_{12}(u), \qquad d_{21}(u^{(1)}) = \widehat{d}_{21}(u), \quad (5.6)$$

$$d_{22}(u^{(1)}) = \widehat{d}_{22}(u) + \psi_1(u^{(1)})\widehat{d}_{11}(u). \quad (5.7)$$

The function $\psi(u)$ is chosen in such a way that it leads to symmetric exchange relations:

$$\widehat{d}_{12}(u)\widehat{d}_{12}(v) = \begin{cases} \widehat{d}_{12}(v)\widehat{d}_{12}(u), & \text{for } \mathfrak{D}_{2|0}, \mathfrak{D}_{0|2} \\ \mathfrak{h}(u,v)\widehat{d}_{12}(v)\widehat{d}_{12}(u), & \text{for } \mathfrak{D}_{1|1}, \end{cases} \quad (5.8)$$

$$\widehat{d}_{11}(u)\widehat{d}_{12}(v) = \mathfrak{f}_1(u,v)\widehat{d}_{12}(v)\widehat{d}_{11}(u) + \mathfrak{g}_1(u,v)\widehat{d}_{12}(u)\widehat{d}_{11}(v) + \mathfrak{h}_1(u,v)\widehat{d}_{12}(u)\widehat{d}_{22}(v), \quad (5.9)$$

$$\widehat{d}_{22}(u)\widehat{d}_{12}(v) = \widetilde{\mathfrak{f}}_2(u,v)\widehat{d}_{12}(v)\widehat{d}_{22}(u) + \widetilde{\mathfrak{g}}_2(u,v)\widehat{d}_{12}(u)\widehat{d}_{22}(v) + \widetilde{\mathfrak{h}}_2(u,v)\widehat{d}_{12}(u)\widehat{d}_{11}(v). \quad (5.10)$$

The explicit form of the functions appearing above is given in appendix B. In the new basis, Ω is still a pseudo-vacuum:

$$\widehat{d}_{11}(u)\Omega = \widehat{\Lambda}_1(u)\Omega = \Lambda_1(u^{(1)})\Omega, \qquad \widehat{d}_{21}(u)\Omega = 0, \quad (5.11)$$

$$\widehat{d}_{22}(u)\Omega = \widehat{\Lambda}_2(u)\Omega = \big(\Lambda_2(u^{(1)}) - \psi_1(u^{(1)})\Lambda_1(u^{(1)})\big)\Omega, \quad (5.12)$$

and we can use the algebraic Bethe ansatz as in the closed chain case. The transfer matrix can be rewritten

$$d(u^{(1)}) = ((-1)^{[1]}m_1 k(u^{(1)}) + (-1)^{[2]}m_2\psi_1(u^{(1)}))\widehat{d}_{11}(u) + (-1)^{[2]}m_2\widehat{d}_{22}(u) \equiv \widehat{d}(u). \quad (5.13)$$

Applying M creation operators $\widehat{d}_{12}(u_j)$ on the pseudo-vacuum we generate a Bethe vector:

$$\Phi(\{u\}) = \widehat{d}_{12}(u_1)\cdots\widehat{d}_{12}(u_M)\Omega. \quad (5.14)$$

Demanding $\Phi(\{u\})$ to be an eigenvector of $\widehat{d}(u)$ leads to a set of algebraic relations on the parameters $u_1, \ldots, u_M$, the so-called Bethe equations. The relation (5.8) between creation operators proves the invariance (up to a function for $\mathfrak{D}_{1|1}$) of the Bethe vector under the

reordering of creation operators. This condition is useful to compute the unwanted terms from the action of $\widehat{d}(u)$ on $\Phi(\{u\})$. We compute the action of $\widehat{d}_{11}(u)$ on $\Phi(\{u\})$,

$$
\widehat{d}_{11}(u)\Phi(\{u\}) = \prod_{k=1}^{M} \mathfrak{f}_1(u, u_k)\widehat{\Lambda}_1(u)\Phi(\{u\})
$$

$$
+ \sum_{k=1}^{M}(M_k(u, \{u\})\widehat{\Lambda}_1(u_k) + N_k(u, \{u\})\widehat{\Lambda}_2(u_k))\Phi_k(u, \{u\}), \tag{5.15}
$$

$$
\Phi_k(u, \{u\}) = \widehat{d}_{12}(u_1)\cdots\widehat{d}_{12}(u_k \to u)\cdots\widehat{d}_{12}(u_M)\Omega,
$$

where the notation $\widehat{d}_{12}(u_k \to u)$ is used to indicate the position of $\widehat{d}_{12}(u)$ in the ordered product. The form of $M_1(u; \{u\})$ and $N_1(u; \{u\})$ is easily computed. The other polynomials $M_k(u; \{u\})$ and $N_k(u; \{u\})$ are then computed using the commutation relation between the $\widehat{d}_{12}(u)$ operators and putting $\widehat{d}_{12}(u_k)$ on the left. We obtain

$$
M_k(u, \{u\}) = \mathfrak{g}_1(u, u_k) \prod_{i \neq k}^{M} \mathfrak{f}_1(u_k, u_i) \qquad \text{and} \qquad N_k(u, \{u\}) = \mathfrak{h}_1(u, u_k) \prod_{i \neq k}^{M} \widetilde{\mathfrak{f}}_2(u_k, u_i).
$$

Similarly, we compute the action of $\widehat{d}_{22}(u)$ on $\Phi(\{u\})$,

$$
\widehat{d}_{22}(u)\Phi(\{u\}) = \left(\prod_{k=1}^{M} \widetilde{\mathfrak{f}}_2(u, u_k)\right)\widehat{\Lambda}_2(u)\Phi(\{u\})
$$

$$
+ \sum_{k=1}^{M}(O_k(u, \{u\})\widehat{\Lambda}_2(u_k) + P_k(u, \{u\})\widehat{\Lambda}_1(u_k))\Phi_k(u, \{u\}), \tag{5.16}
$$

$$
O_k(u, \{u\}) = \widetilde{\mathfrak{g}}_2(u, u_k) \prod_{i \neq k}^{M} \widetilde{\mathfrak{f}}_1(u_k, u_i) \quad \text{and} \quad P_k(u, \{u\}) = \widetilde{\mathfrak{h}}_2(u, u_k) \prod_{i \neq k}^{M} \mathfrak{f}_1(u_k, u_i). \tag{5.17}
$$

Demanding that $\Phi(\{u\})$ be an eigenvector of $\widehat{d}(u)$ corresponds to the cancelling of the so-called 'unwanted terms' carried by the vectors $\Phi_k(u, \{u\})$. In this way, we get the Bethe equations:

$$
\frac{\widehat{\Lambda}_1(u_k)}{\widehat{\Lambda}_2(u_k)} = \chi_1(u_k) \prod_{i \neq k}^{M} \frac{\widetilde{\mathfrak{f}}_2(u_k, u_i)}{\mathfrak{f}_1(u_k, u_i)}, \qquad k = 1, \ldots, M. \tag{5.18}
$$

Remark that the r.h.s. depends only on the structure constants of the (super)algebra under consideration, while the l.h.s. encodes the representations entering the spin chain. Then, the eigenvalues of the transfer matrix read:

$$
\widehat{d}(u)\Phi(\{u\}) = \widehat{\Lambda}(u; \{u\})\Phi(\{u\}), \tag{5.19}
$$

$$
\widehat{\Lambda}(u; \{u\}) = ((-1)^{[1]}m_1 k(u^{(1)}) + (-1)^{[2]}m_2 \psi_1(u^{(1)}))\widehat{\Lambda}_1(u) \prod_{k=1}^{M} \mathfrak{f}_1(u, u_k)
$$

$$
+ (-1)^{[2]}m_2\widehat{\Lambda}_2(u) \prod_{k=1}^{M} \widetilde{\mathfrak{f}}_2(u, u_k). \tag{5.20}
$$

Note that Bethe equations correspond to the vanishing of the residue of $\Lambda(u; \{u\})$. This is the tool used in analytical Bethe ansatz [12] to obtain Bethe equations, see e.g. [13, 15].

6. Nested Bethe ansatz

6.1. Preliminaries

The method, called the nested Bethe ansatz (NBA), consists in a recurrent application of the ABA to express higher rank solutions using the lower ones. It has been introduced in [6] for the periodic case. The same method can be used for the boundary case. In this way, we can compute the eigenvalues, eigenvectors and Bethe equations of the $\mathfrak{D}_{m|n}$ model from those of the $\mathfrak{D}_2$ or $\mathfrak{D}_{1|1}$ model. Although we are in a (tensor product of) representation(s) of $\mathfrak{D}_{m|n}$, we will loosely keep writing $d_{ij}(u)$ the representation of the operators $d_{ij}(u)$, assuming that the reader will understand that when $d_{ij}(u)$ applies to the highest weight Ω, it is in fact its (matricial) representation that is used. Another way to understand this method in an algebraic way is to work in the coset of the $\mathfrak{D}_{m|n}$ algebra by the left ideal $\mathcal{I}_{m+n}$.

We consider now the open case with general diagonal boundary condition (3.9) for $K^-(u)$, and $K^+(u)$ of the form:

$$K^+(u) = MK(u), \quad \text{with} \quad K(u) = \begin{cases} \mathbb{I} & \text{for} \quad a_+ = 0 \\ \operatorname{diag}(k(u), 1, \dots, 1) & \text{for} \quad a_+ = 1, \end{cases}$$

where the function $k(u)$ is defined in (5.4). The matrix $K^+(u) = MK(u)$ is the only solution we can use to perform the NBA up to the end (see remarks 6.1 and 6.2 below). We decompose the monodromy matrix in the following form (in the End($\mathbb{C}^{m+n}$) auxiliary space),

$$D(u) = \begin{pmatrix} d_{11}(u) & B^{(1)}(u) \\ C^{(1)}(u) & D^{(2)}(u) \end{pmatrix}, \tag{6.1}$$

where $B^{(1)}(u)$ (resp. $C^{(1)}(u)$) is a row (resp. column) vector of $\mathbb{C}^{m+n-1}$, and $D^{(2)}(u)$ is a matrix of End($\mathbb{C}^{m+n-1}$).

Then, $D^{(2)}(u)$ is itself decomposed in the same way, and more generally, for a given k in $\{1, \dots, m+n-2\}$, we gather the generators $d_{kj}(u)$, (resp. $d_{jk}(u)$) $j = k+1, \dots, n+m$, in a row (resp. column) vector of $\mathbb{C}^{m+n-k}$ and $d_{ij}(u)$, $i, j \geqslant k$, into a matrix of End($\mathbb{C}^{m+n-k}$):

$$B^{(k)}(u) = \sum_{j=k+1}^{m+n} e_j^t \otimes d_{kj}(u) \qquad \text{and} \qquad C^{(k)}(u) = \sum_{j=k+1}^{m+n} e_j \otimes d_{jk}(u), \tag{6.2}$$

$$D^{(k+1)}(u) = \sum_{i,j=k+1}^{m+n} E_{ij} \otimes d_{ij}(u), \tag{6.3}$$

$$D^{(k)}(u) = \begin{pmatrix} d_{kk}(u) & B^{(k)}(u) \\ C^{(k)}(u) & D^{(k+1)}(u) \end{pmatrix}. \tag{6.4}$$

We decompose the transfer matrix in the same way:

$$\begin{aligned} d(u) &= d^{(1)}(u) = (-1)^{[1]} m_1 k(u) d_{11}(u) + d^{(2)}(u), \\ d^{(k)}(u) &= \operatorname{str}\left(M_a^{(k)} D^{(k)}(u)\right) = (-1)^{[k]} m_k d_{kk}(u) + d^{(k+1)}(u), \\ M^{(k)} &= \mathbb{I}^{(k)} M \mathbb{I}^{(k)}. \end{aligned} \tag{6.5}$$
$$\tag{6.6}$$

At each step of the recursion, we make a transformation of the operator and a shift of the spectral parameter:

$$\begin{aligned} d_{kk}(u^{(k)}) &= \widehat{d}_{kk}(u), \qquad B_a^{(k)}(u^{(k)}) = \widehat{B}_a^{(k)}(u), \\ D_a^{(k+1)}(u^{(k)}) &= \widehat{D}_a^{(k+1)}(u) + \psi_k(u^{(k)}) \mathbb{I}_a^{(k+1)} \otimes \widehat{d}_{kk}(u). \end{aligned} \tag{6.7}$$

The commutation relations for these operators remain similar for each k:

$$\widehat{B}_a^{(k)}(u)\widehat{B}_b^{(k)}(v) = (-1)^{[k]}\frac{\mathfrak{a}_{k+1}(u^{(k)}, v^{(k)})}{\mathfrak{a}_k(u^{(k)}, v^{(k)})}\widehat{B}_b^{(k)}(v)\widehat{B}_a^{(k)}(u)\mathbb{R}_{ba}^{(k+1)}(u, v), \tag{6.8}$$

$$\widehat{d}_{kk}(u)\widehat{B}_b^{(k)}(v) = \mathfrak{f}_k(u, v)\widehat{B}_b(v)\widehat{d}_{kk}(u) + \mathfrak{g}_k(u, v)\widehat{B}_b^{(k)}(u)\widehat{d}_{kk}(v)$$

$$+ \frac{\mathfrak{h}_k(u, v)}{\mathfrak{e}_{k+1}(v)}\widehat{B}_b^{(k)}(u)\mathrm{str}_a\big(M_a^{(k+1)}\overline{\mathbb{R}}_{ab}^{(k+1)}(v, v)\widehat{D}_a^{(k+1)}(v)\mathbb{R}_{ba}^{(k+1)}(v, v)\big), \tag{6.9}$$

$$\mathrm{str}_a\big(M_a^{(k+1)}\widehat{D}_a^{(k+1)}(u)\big)\widehat{B}_b^{(k)}(v) = \widetilde{\mathfrak{h}}_{k+1}(u, v)\mathfrak{e}_{k+1}(u)\widehat{B}_b^{(k)}(u)\widehat{d}_{kk}(v)$$

$$+ \frac{\widetilde{\mathfrak{g}}_{k+1}(u, v)\mathfrak{e}_{k+1}(u)}{\mathfrak{e}_{k+1}(v)}\widehat{B}_b^{(k)}(u)\,\mathrm{str}_a\big(M_a^{(k+1)}\overline{\mathbb{R}}_{ab}^{(k+1)}(v, v)\widehat{D}_a^{(k+1)}(v)\mathbb{R}_{ba}^{(k+1)}(v, v)\big)$$

$$+ \widetilde{\mathfrak{f}}_{k+1}(u, v)\widehat{B}_b^{(k)}(v)\,\mathrm{str}_a\big(M_a^{(k+1)}\overline{\mathbb{R}}_{ab}^{(k+1)}(u, v)\widehat{D}_a^{(k+1)}(u)\mathbb{R}_{ba}^{(k+1)}(u, v)\big). \tag{6.10}$$

Remark 6.1. The commutation relations (6.9) and (6.10) impose the restriction on the $K^+(u)$ matrix. The direct use of the reflection equation leads to a matrix $\overline{\mathbb{R}}_{ab}^{(k+1)}(u, u)$ in (6.9) and (6.10). The change $\overline{\mathbb{R}}_{ab}^{(k+1)}(u, u) \to \overline{\mathbb{R}}_{ab}^{(k+1)}(v, v)$ in the commutation relation is allowed by equality (B.5) which shows that the dependence in u is a scalar function. If the $K^+(u)$ matrix is not from a *NABA couple*, equation (B.5) cannot be used to get (6.9) and (6.10) in their present form. Without this form, the nesting cannot be performed (see also remark 6.2).

At each step $k = 1, \ldots, \mathfrak{m} + \mathfrak{n} - 1$ of the nesting, we will introduce a family of Bethe parameters u_{kj}, $j = 1, \ldots, M_k$, the number M_k of these parameters being a free integer. The partial unions of these families will be denoted as

$$\{u_\ell\} = \bigcup_{i=1}^{\ell}\{u_{ij}, j = 1, \ldots, M_i\}, \tag{6.11}$$

so that the whole family of Bethe parameters is $\{u\} = \{u_{\mathfrak{m}+\mathfrak{n}-1}\}$.

6.2. First step of the construction

From the definition of the highest weight, $C^{(1)}(u)$ annihilates the pseudo-vacuum Ω and we can use $B^{(1)}(u)$ as a creation operator. However, since $B^{(1)}(u)$ contains only $d_{1j}(u)$ operators, it is clear that we need to act on several vectors to describe the whole representation with highest weight Ω. The NBA spirit is to construct these different vectors as Bethe vectors of a $\mathfrak{D}_{\mathfrak{m}-1|\mathfrak{n}}$ chain that is related to the chain we start with.

More generally, at each step k corresponding to the decomposition (6.4) of the monodromy matrix and to the transformation of the operator of the corresponding algebra $\mathfrak{D}_{\mathfrak{m}-k|\mathfrak{n}}$, we use (a suitable refinement of) $B^{(k)}(u)$ as a creation operator acting on a set of (to be defined) vectors. These vectors are constructed as Bethe vectors of a $\mathfrak{D}_{\mathfrak{m}-k-1|\mathfrak{n}}$ chain.

At the first step of the recursion, the Bethe vectors have the form

$$\Phi(\{u^{(1)}\}) = \widehat{B}_{a_1^1}^{(1)}(u_{11})\cdots\widehat{B}_{a_{M_1}^1}^{(1)}\big(u_{1M_1}\big)\widehat{F}_{a_1^1\ldots a_{M_1}^1}^{(1)}(\{u\})\Omega, \tag{6.12}$$

$$\widehat{F}_{a_1^1\ldots a_{M_1}^1}^{(1)}(\{u\}) \in (\mathbb{C}^{\mathfrak{m}-1|\mathfrak{n}})^{\otimes M_1} \otimes \mathfrak{D}_{\mathfrak{m}-1|\mathfrak{n}}, \tag{6.13}$$

where $\widehat{F}_{a_1^1\ldots a_{M_1}^1}^{(1)}(\{u\})$ is built from operators $\widehat{d}_{ij}(u)$, $2 \leqslant i \leqslant j \leqslant \mathfrak{m} + \mathfrak{n}$ only. Since $\widehat{B}^{(1)}(u)$ belongs to $\mathbb{C}^{\mathfrak{m}-1|\mathfrak{n}} \otimes \mathfrak{D}_{\mathfrak{m}|\mathfrak{n}}$, we have introduced in the construction M_1 additional auxiliary

spaces (labelled $a_1^1, \ldots, a_{M_1}^1$) that are also carried by $\widehat{F}^{(1)}_{a_1^1 \ldots a_{M_1}^1}(\{u\})$. These new auxiliary spaces take care of the linear combination one has to do between the different generators $\widehat{d}_{1j}(u)$, $j = 2, \ldots, \mathfrak{m} + \mathfrak{n}$, that enter into the construction.

Since $F^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})$ is built up from operators $\widehat{d}_{ij}(u)$, $2 \leqslant i \leqslant j \leqslant \mathfrak{m} + \mathfrak{n}$, it obeys the relation (proven in a more general context in theorem 3.1):

$$\widehat{d}_{11}(u)\widehat{F}^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\Omega = \widehat{\Lambda}_1(u)\widehat{F}^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\Omega. \tag{6.14}$$

The transfer matrix is decomposed into

$$d(u^{(1)}) = \widetilde{\mathfrak{m}}_1(u)\widehat{d}_{11}(u) + \widehat{d}^{(2)}(u) \qquad \text{with} \qquad \widehat{d}^{(2)}(u) = \mathrm{str}_a\big(M_a^{(2)}\widehat{D}_a^{(2)}(u)\big) \tag{6.15}$$

$$\widetilde{\mathfrak{m}}_1(u) = (-1)^{[1]}m_1 k(u) + \mathrm{str}(M^{(2)})\psi_1(u^{(1)}). \tag{6.16}$$

The action of $\widehat{d}_{11}(u)$ on $\Phi(\{u^{(1)}\})$ takes the form

$$\widehat{d}_{11}(u)\Phi(\{u\}) = \widehat{\Lambda}_1(u)\prod_{i=1}^{M_1}\mathfrak{f}_1(u, u_{1i})\Phi(\{u\}) + \sum_{j=1}^{M_1}M_j(u; \{u_1\})\widehat{\Lambda}_1(u_{1j})\widehat{\Phi}_j(u, \{u\})$$

$$+ \sum_{j=1}^{M_1}N_j(u; \{u_1\})\widehat{B}^{(1)}_{a_1}(u_{11})\cdots\widehat{B}^{(1)}_{a_j}(u)\cdots\widehat{B}^{(1)}_{a_{M_1}}(u_{1M_1})$$

$$\times \widetilde{d}^{(2)}(u_{1j}; \{u_1\})\widehat{F}^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\Omega, \tag{6.17}$$

with

$$M_j(u; \{u_1\}) = \mathfrak{g}_1(u, u_{1j})\prod_{i \neq j}^{M_1}\mathfrak{f}_1(u_{1i}, u_{1j}) \qquad \text{and} \qquad N_j(u; \{u\}) = \frac{\mathfrak{h}_1(u, u_{1j})}{\mathfrak{e}_2(u_{1j})}\prod_{i \neq j}^{M_1}\widetilde{\mathfrak{f}}_2(u_{1i}, u_{1j}).$$

The action of $\widehat{d}^{(2)}(u)$ on $\Phi(\{u\})$ takes the form

$$\widehat{d}^{(2)}(u)\Phi(\{u^{(1)}\}) = \prod_{j=1}^{M_1}\widetilde{\mathfrak{f}}_2(u, u_{1j})B^{(1)}_{a_1}(u_{11})\cdots\widehat{B}^{(1)}_{a_{M_1}}(u_{1M_1})\widetilde{d}^{(2)}(u; \{u_1\})\widehat{F}^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\Omega$$

$$+ \sum_{j=1}^{M_1}P_j(u; \{u_1\})\widehat{B}^{(1)}_{a_1}(u_{11})\cdots B^{(1)}_{a_j}(u)\cdots\widehat{B}^{(1)}_{a_{M_1}}(u_{1M_1})\widetilde{d}^{(2)}(u_{1j}; \{u_1\})$$

$$\times \widehat{F}^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\Omega + \sum_{j=1}^{M_1}Q_j(u; \{u_1\})\widehat{\Lambda}_1(u_{1j})\Phi_j(\{u\}), \tag{6.18}$$

with

$$P_j(u; \{u_1\}) = \frac{\widetilde{\mathfrak{g}}_2(u, u_{1i})\mathfrak{e}_2(u)}{\mathfrak{e}_2(u_{1i})}\prod_{i \neq j}^{M_1}\widetilde{\mathfrak{f}}_2(u_{1j}, u_{1i}),$$

$$Q_j(u; \{u_1\}) = \widetilde{\mathfrak{h}}_2(u, u_{1i})\mathfrak{e}_2(u)\prod_{i \neq j}^{M_1}\mathfrak{f}_1(u_{1j}, u_{1i}),$$

where $\Phi_j(\{u\})$ is deduced from $\Phi(\{u\})$ by the change $u_{1j} \rightarrow u$. These expressions are computed as has been done in section 5: $N_1(u; \{u_1\})$, $M_1(u; \{u_1\})$, $P_1(u; \{u_1\})$ and $Q_1(u; \{u_1\})$ are easy to compute; the other terms are obtained through a reordering of the

22

operators $\widehat{B}^{(1)}(u_{1j})$, using the reordering lemma 6.1 and the Yang–Baxter equation. We also used the notation:

$$\widetilde{d}^{(2)}(u; \{u_1\}) = \text{str}_a \left(M_a^{(2)} \prod_{j=1}^{\overrightarrow{M_1}} \overline{\mathbb{R}}_{a,a_j}^{(2)}(u, u_{1j}) \widehat{D}_a^{(2)}(u) \prod_{j=1}^{\overleftarrow{M_1}} \mathbb{R}_{a_j,a}^{(2)}(u, u_{1j}) \right). \tag{6.19}$$

Remark 6.2. The fact that the wanted and unwanted terms contain the same operator $\widetilde{d}^{(2)}$ (but at different values u and u_{1j}) allows one to continue the nesting. In this way, the diagonalization of this operator allows us at the same time to compute the eigenvalue *and* to show that the unwanted terms cancel (when the Bethe ansatz equations are obeyed). The apparition of this operator in the unwanted terms is directly related to the present form of the commutation relations (6.9) and (6.10), see remark 6.1. Hence the need of a NABA couple to perform the nesting.

As already mentioned, the calculation makes a new transfer matrix $\widetilde{d}^{(2)}(u; \{u_1\})$ appear corresponding to a $\mathfrak{D}_{m-1|n}$ chain with $L + M_1$ sites, the M_1 additional sites corresponding to fundamental representations of $\mathfrak{D}_{m-1|n}$. This interpretation is supported by theorem 3.1 which ensures that $\widetilde{D}_a^{(2)}(u; \{u_1\})$ generates $\mathfrak{D}_{m-1|n}$, and that $\widetilde{d}^{(2)}(u; \{u_1\})$ is indeed the transfer matrix of an integrable spin chain. Then, if we assume that $F_{a_1 \dots a_{M_1}}^{(1)}(\{u\})\Omega$ is an eigenvector of this new transfer matrix,

$$\widetilde{d}^{(2)}(u; \{u_1\})\widehat{F}_{a_1 \dots a_{M_1}}^{(1)}(\{u\})\Omega = \widetilde{\Gamma}^{(2)}(u)\widehat{F}_{a_1 \dots a_{M_1}}^{(1)}(\{u\})\Omega, \tag{6.20}$$

we deduce

$$\widehat{d}_{11}(u)\Phi(\{u\}) = \widehat{\Lambda}_1(u) \prod_{i=1}^{M_1} \mathfrak{f}_1(u, u_{1i}) \Phi(\{u\})$$

$$+ \sum_{j=1}^{M_1} (M_j(u; \{u_1\})\widehat{\Lambda}_1(u_{1j}) + N_j(u; \{u_1\})\widetilde{\Gamma}^{(2)}(u_{1j}))\Phi_j(\{u\}), \tag{6.21}$$

$$\widehat{d}^{(2)}(u)\Phi(\{u\}) = \widetilde{\Gamma}^{(2)}(u) \prod_{j=1}^{M_1} \widetilde{\mathfrak{f}}_2(u, u_{1j}) \Phi(\{u\})$$

$$+ \sum_{j=1}^{M_1} (P_j(u; \{u_1\})\widetilde{\Gamma}^{(2)}(u_{1j}) + Q_j(u; \{u_1\})\widehat{\Lambda}_1(u_{1j}))\Phi_j(\{u\}). \tag{6.22}$$

Gathering these relations together, we get a first expression of the action of $d(u)$ on $\Phi(\{u\})$. When we cancel in this expression the unwanted terms (carried by $\Phi_j(\{u\})$), we get the first system of Bethe equations:

$$\frac{\widehat{\Lambda}_1(u_{1j})}{\widetilde{\Gamma}^{(2)}(u_{1j})} = \frac{\chi_1(u_{1j})}{\mathfrak{e}_2(u_{1j})} \prod_{i \neq j}^{M_1} \frac{\widetilde{\mathfrak{f}}_2(u_{1j}, u_{1i})}{\mathfrak{f}_1(u_{1j}, u_{1i})}. \tag{6.23}$$

We also get a first expression of the transfer matrix eigenvalue:

$$\widehat{d}^{(1)}(u)\Phi(\{u\}) = \left(\widetilde{\mathfrak{m}}_1(u)\widehat{\Lambda}_1(u) \prod_{j=1}^{M_1} \mathfrak{f}_1(u, u_{1j}) + \widetilde{\Gamma}^{(2)}(u) \prod_{j=1}^{M_1} \widetilde{\mathfrak{f}}_1(u, u_{1j}) \right) \Phi(\{u\}). \tag{6.24}$$

In the above relations, everything is known *but* the eigenvalue $\widetilde{\Gamma}^{(2)}(u)$, introduced in (6.20), and the explicit form of $\widehat{F}_{a_1^1 \dots a_{M_1}^1}^{(1)}(\{u\})$ ensuring that (6.20) is indeed satisfied.

Thus, at the end of this first recursion step, we have 'reduced' the problem of computing an eigenvector $\Phi(\{u\})$ for the transfer matrix $d(u)$ of a $\mathfrak{D}_{m|n}$ chain with L sites to the problem of computing an eigenvector $\Phi^{(1)}(\{u\}) = \widehat{F}^{(1)}_{a_1^1 \ldots a_{M_1}^1}(\{u\})\Omega$ for the transfer matrix $\widetilde{d}^{(2)}(u; \{u_1\})$ of a $\mathfrak{D}_{m-1|n}$ chain with $L + M_1$ sites.

Remark 6.3 (change of notation). To avoid too complicated a notation for the second step, we need to slightly change the notation at the end of the first step. First, we rename the hatted operators $\widehat{X}(u) \to X(u)$, although they still have the spectral parameter shift and the operator transformation coming from the first step. Second, we omit the tilde on operators, $\widetilde{X}(u) \to X(u)$, keeping in mind that the new operators $X(u)$ have M_1 sites more than the one of the previous step. In this way, we will be able to re-use the 'hatted' and 'tilded' notations for the transformations used in the second step.

This will be the general approach at each step: at the end of step k, we will perform a change of notation, suppressing the hats and tildes on operators, to use them again in step $k + 1$.

It remains to single out the highest weights corresponding to the fundamental representations carried by the new sites. This is done in the following way

$$\Phi^{(1)}(\{u\}) = F^{(1)}_{a_1 \ldots a_{M_1}}(\{u\})\Omega,$$

$$\Phi^{(1)}(\{u\}) = B^{(2)}_{a_1^2}(u_{21}; \{u_1\}) \cdots B^{(2)}_{a_{M_2}^2}(u_{2M_2}; \{u_1\}) F^{(2)}_{a_1^2 \ldots a_{M^2}^2}(\{u\})\Omega^{(2)}, \quad (6.25)$$

$$\Omega^{(2)} = \left(e_1^{(1)}\right)^{\otimes M_1} \otimes \Omega, \quad (6.26)$$

where $e_1^{(1)} = (1, 0, \ldots, 0)^t \in \mathbb{C}^{m-1|n}$ and $F^{(2)}_{a_1^2 \ldots a_{M_2}^2}(\{u\})$ is built on operators $d_{ij}(u^{(2)}; \{u_1\})$, with $j \geqslant i > 2$. The operators $B^{(2)}(u; \{u_1\})$ play the role, for the $\mathfrak{D}_{m-1|n}$ chain of length $L + M_1$, of the operators $B^{(1)}(u)$ for the $\mathfrak{D}_{m|n}$ chain of length L. Explicitly, they are obtained from the decomposition (6.4) of the monodromy matrix.

6.3. General step

More generally, the step k starts with the problem

$$d^{(k)}(u; \{u_{k-1}\})\Phi^{(k-1)}(\{u\}) = \Gamma^{(k)}(u)\Phi^{(k-1)}(\{u\}), \quad (6.27)$$

where $d^{(k)}(u; \{u_{k-1}\}) = \operatorname{str}(M^{(k)}D^{(k)}(u; \{u_{k-1}\}))$ is the transfer matrix of a $\mathfrak{D}_{m-k-1|n}$ spin chain of length $L + \sum_{j=1}^{k-1} M_j$ (obtained from the previous step). We recall that hats and tildes have been suppressed, according to remark 6.3, including for the function $\Gamma^{(k)}(u)$. We define

$$\Phi^{(k-1)}(\{u\}) = F^{(k-1)}_{a_1^{k-1} \ldots a_{M_{k-1}}^{k-1}}(\{u\})\Omega^{(k-1)} = \mathbb{B}^{(k)}(\{u_k\}) F^{(k)}_{a_1^k \ldots a_{M_k}^k}(\{u\})\Omega^{(k)}, \quad (6.28)$$

$$\Omega^{(k)} = \left(e_1^{(k-1)}\right)^{\otimes M_{k-1}} \otimes \Omega^{(k-1)}, \quad (6.29)$$

with $e_1^{(k)} = (1, 0, \ldots, 0)^t \in \mathbb{C}^{m-k|n}$. We have introduced

$$\mathbb{B}^{(k)}(\{u_k\}) = B^{(k)}_{a_1^k}(u_{k1}; \{u_{k-1}\}) \cdots B^{(k)}_{a_{M_k}^k}(u_{kM_k}; \{u_{k-1}\}), \quad (6.30)$$

where the operators are extracted from the monodromy matrix, see equation (6.4).

Remark 6.4. In (6.30), we have indicated only the auxiliary spaces a_j^k, $j = 1, \ldots, M_k$. In fact, since $D^{(k)}(u; \{u_{k-1}\})$ is viewed as the monodromy matrix of a spin chain of length

$L + \sum_{j=1}^{k-1} M_j$, the other spaces a_j^ℓ, $j = 1, \ldots, M_\ell$, $\ell < k$, are now quantum spaces. Thus, they do not appear explicitly in $D^{(k)}$, as the sites of the original spin chain, but obviously this monodromy matrix (and its components) does depend on all these spaces.

We extract from $d^{(k)}(u; \{u_{k-1}\})$ the component $d_{kk}(u; \{u_{k-1}\})$:

$$d^{(k)}(u; \{u_{k-1}\}) = (-1)^{[k]} m_k d_{kk}(u; \{u_{k-1}\}) + \text{str}(M^{(k+1)} D^{(k+1)}(u; \{u_{k-1}\})). \tag{6.31}$$

Now we must transform the operator:

$$\begin{aligned} d_{kk}(u^{(k)}) &= \widehat{d}_{kk}(u), & B_a^{(k)}(u^{(k)}) &= \widehat{B}_a^{(k)}(u), \\ D^{(k+1)}(u^{(k)}) &= \widehat{D}^{(k+1)}(u) + \psi_k(u^{(k)})\widehat{d}_{kk}(u)\mathbb{I}^{(k)}. \end{aligned} \tag{6.32}$$

The transfer matrix $d^{(k)}(u^{(k)}; \{u_{k-1}\})$ is rewritten as

$$\widehat{d}^{(k)}(u; \{u_{k-1}\}) = \widetilde{\mathfrak{m}}_k(u)\widehat{d}_{kk}(u; \{u_{k-1}\}) + \text{str}(M^{(k+1)}\widehat{D}^{(k+1)}(u; \{u_{k-1}\})), \tag{6.33}$$

$$\widetilde{\mathfrak{m}}_k(u) = (-1)^{[k]} m_k + \text{str}(M^{(k)})\psi_k(u^{(k)}), \tag{6.34}$$

and the Bethe vector:

$$\Phi^{(k-1)}(\{u\}) = \widehat{\mathbb{B}}^{(k)}(\{u_k\})\widehat{F}^{(k)}_{a_1^k \ldots a_{M_k}^k}(\{u\})\Omega^{(k)}, \tag{6.35}$$

$$\widehat{\mathbb{B}}^{(k)}(\{u_k\}) = \widehat{B}^{(k)}_{a_1^k}(u_{k1}; \{u_{k-1}\}) \cdots \widehat{B}^{(k)}_{a_{M_k}^k}(u_{kM_k}; \{u_{k-1}\}). \tag{6.36}$$

Now we can compute the action of the transfer matrix on this vector. We first commute $\widehat{d}_{kk}(u; \{u_{k-1}\})$ and $\widehat{d}^{(k+1)}(u; \{u_{k-1}\}) = \text{str}(M^{(k)}\widehat{D}^{(k+1)}(u; \{u_{k-1}\}))$ with the operator $\widehat{\mathbb{B}}^{(k)}(\{u_k\})$:

$$\begin{aligned} \widehat{d}_{kk}(u; \{u_{k-1}\})\Phi^{(k-1)}(\{u^{(k)}\}) &= \prod_{j=1}^{M_k} \mathfrak{f}_k(u, u_{kj})\widehat{\mathbb{B}}^{(k)}(\{u_k\})\widehat{d}_{kk}(u; \{u_{k-1}\})\Phi^{(k)}(\{u\}) \\ &+ \sum_{j=1}^{M_k} M_j(u; \{u_{k-1}\})\widehat{\mathbb{B}}^{(k)}_j(u; \{u_k\})\widehat{d}_{kk}(u_{kj}; \{u_{k-1}\})\Phi^{(k)}(\{u\}) \\ &+ \sum_{j=1}^{M_k} N_j(u; \{u_{k-1}\})\widehat{\mathbb{B}}^{(k)}_j(u; \{u_k\})\widetilde{d}^{(k+1)}(u_{kj}; \{u_k\})\Phi^{(k)}(\{u\}), \end{aligned} \tag{6.37}$$

$$\begin{aligned} \widehat{d}^{(k+1)}(u; \{u_{k-1}\})\Phi^{(k-1)}(\{u\}) &= \prod_{j=1}^{M_k} \widetilde{\mathfrak{f}}_{k+1}(u, u_{kj})\widehat{\mathbb{B}}^{(k)}(\{u_k\})\widetilde{d}^{(k+1)}(u; \{u_k\})\Phi^{(k)}(\{u\}) \\ &+ \sum_{j=1}^{M_k} P_j(u; \{u_{k-1}\})\widehat{\mathbb{B}}^{(k)}_j(u; \{u_k\})\widetilde{d}^{(k+1)}(u_{kj}; \{u_k\})\Phi^{(k)}(\{u\}) \\ &+ \sum_{j=1}^{M_k} Q_j(u; \{u_{k-1}\})\widehat{\mathbb{B}}^{(k)}_j(u; \{u_k\})\widehat{d}_{kk}(u_{kj}; \{u_{k-1}\})\Phi^{(k)}(\{u\}), \end{aligned} \tag{6.38}$$

where we have introduced:

$$\Phi^{(k)}(\{u\}) = \widehat{F}^{(k)}_{a_1^k \ldots a_{M_k}^k}(\{u\})\Omega^{(k)},$$

$$\widetilde{d}^{(k+1)}(u; \{u_k\}) = \text{str}_a\left(M_a^{(k+1)}\left(\prod_{j=1}^{\overrightarrow{M_k}} \overline{\mathbb{R}}^{(k+1)}_{aa_j^k}(u, u_{kj})\right) \widehat{D}_a^{(k+1)}(u; \{u_{k-1}\})\left(\prod_{j=1}^{\overleftarrow{M_k}} \mathbb{R}^{(k+1)}_{a_j^k a}(u, u_{kj})\right)\right).$$

The functions M_j, N_j, P_j and Q_j are the same as in the first step (section 6.2) but with indices $1 \to k$ on functions and Bethe roots. We use the following reordering lemma:

Lemma 6.1. *For each $k = 1, \ldots, \mathfrak{m} + \mathfrak{n} - 1$ and $j = 1, \ldots, M_k$, we have*

$$\widehat{\mathbb{B}}^{(k)}(\{u_k\}) = \widehat{B}_j^{(k)}(u_{kj})\widehat{B}_1^{(k)}(u_{k1}) \cdots \widehat{B}_{j-1}^{(k)}(u_{k,j-1})\widehat{B}_{j+1}^{(k)}(u_{k,j+1}) \cdots \widehat{B}_{M_k}^{(k)}(u_{kM_k})$$

$$\times \overrightarrow{\prod_{i=1}^{j-1}}(-1)^{[j]}\frac{\mathfrak{a}_{j+1}\big(u_{ki}^{(k)}, u_{kj}^{(k)}\big)}{\mathfrak{a}_j\big(u_{ki}^{(k)}, u_{kj}^{(k)}\big)}\mathbb{R}_{ji}^{(k+1)}(u_{ki}, u_{kj}), \tag{6.39}$$

where the dependence in $\{u_{k-1}\}$ has been omitted in $\widehat{B}_p^{(k)}$.

Proof. Direct calculation using the commutation relations (6.8)–(6.10).   □

We now compute the action of $\widehat{d}_{kk}(u; \{u_{k-1}\})$ and $\widetilde{d}^{(k+1)}(u; \{u_k\})$ on $\widehat{F}^{(k)}(\{u\})\Omega^{(k)}$. These actions follow from theorem 3.1. For $\widehat{d}_{kk}(u; \{u_{k-1}\})$ we have

$$\widehat{d}_{kk}(u; \{u_{k-1}\})\widehat{F}^{(k)}(\{u\})\Omega^{(k)} = \widehat{\Lambda}_k(u; \{u_{k-1}\})\widehat{F}^{(k)}(\{u\})\Omega^{(k)}. \tag{6.40}$$

It remains to do the same for $\widetilde{d}^{(k+1)}(u; \{u_k\})$. It corresponds to a new monodromy matrix

$$\widetilde{D}_a^{(k+1)}(u; \{u_k\}) = \overrightarrow{\prod_{j=1}^{M_k}}\overline{\mathbb{R}}_{aa_j^k}^{(k+1)}(u, u_{kj})\widehat{D}_a^{(k+1)}(u; \{u_{k-1}\})\overleftarrow{\prod_{j=1}^{M_k}}\mathbb{R}_{a_j^k a}^{(k+1)}(u, u_{kj}). \tag{6.41}$$

It also satisfies the reflection equation, see theorem 3.1, so that the problem is integrable, and defines a $\mathfrak{D}_{\mathfrak{m}-k|\mathfrak{n}}$ spin chain, with $L + \sum_{j=1}^k M_j$ sites.

We get a new eigenvalue problem:

$$\widetilde{d}^{(k+1)}(u; \{u_k\})\Phi^{(k)}(\{u\}) = \widetilde{\Gamma}^{(k+1)}(u)\Phi^{(k)}(\{u\}). \tag{6.42}$$

Assuming the form (6.42), we can show, following the same lines as in the first step, that $\Phi^{(k-1)}(\{u\})$ is a transfer matrix eigenvector provided the kth system of Bethe equations,

$$\frac{\widehat{\Lambda}_k(u_{kj}; \{u_{k-1}\})}{\widetilde{\Gamma}^{(k+1)}(u_{kj}; \{u_k\})} = \frac{\chi_k(u_{kj})}{\mathfrak{e}_{k+1}(u_{kj})}\prod_{i\neq j}^{M_k}\frac{\widetilde{\mathfrak{f}}_{k+1}(u_{kj}, u_{ki})}{\mathfrak{f}_k(u_{kj}, u_{ki})}, \tag{6.43}$$

is obeyed. We also get an expression for $\widehat{\Gamma}^{(k)}(u)$, the eigenvalue of $\widehat{d}^{(k)}(u)$:

$$\widehat{\Gamma}^{(k)}(u) = \widetilde{\mathfrak{m}}_k(u)\prod_{j=1}^{M_k}\mathfrak{f}_k(u, u_{kj})\widehat{\Lambda}_k(u; \{u_{k-1}\}) + \prod_{j=1}^{M_k}\widetilde{\mathfrak{f}}_{k+1}(u, u_{kj})\widetilde{\Gamma}^{(k+1)}(u; \{u_k\}). \tag{6.44}$$

6.4. End of induction

To end the recursion, we use the $\mathfrak{m} + \mathfrak{n} = 2$ case and remark that

$$\widehat{\Gamma}^{(\mathfrak{n}+\mathfrak{m})}(u) = \widehat{\Lambda}_{\mathfrak{n}+\mathfrak{m}}(u, \{u_{\mathfrak{m}+\mathfrak{n}-2}\}). \tag{6.45}$$

Using the shift notation, $u^{(k\ldots l)} = (\cdots(u^{(k)})^{(k+1)}\cdots)^{(l)}$, for $k \leqslant l$, we deduce from (6.45) that $\widehat{\Gamma}$ is expressed in terms of $\widehat{\Lambda}$:

$$\widehat{\Gamma}^{(k+1)}(u^{(k+2\ldots\mathfrak{n}+\mathfrak{m}-1)}) = \widetilde{\mathfrak{m}}_{k+1}(u^{(k+2\ldots\mathfrak{n}+\mathfrak{m}-1)})\widehat{\Lambda}_{k+1}(u^{(k+2\ldots\mathfrak{n}+\mathfrak{m}-1)}; \{u_k\})\mathcal{F}_{k+1}(u)$$

$$+ \sum_{\ell=k+2}^{\mathfrak{m}+\mathfrak{n}-1}\widetilde{\mathfrak{m}}_\ell(u^{(\ell+1\ldots\mathfrak{n}+\mathfrak{m}-1)})\widehat{\Lambda}_\ell(u^{(\ell+1\ldots\mathfrak{n}+\mathfrak{m}-1)}; \{u_{\ell-1}\})\mathcal{F}_\ell(u)\prod_{p=k+1}^{\ell-1}\widetilde{\mathcal{F}}_p(u)$$

$$+ (-1)^{[\mathfrak{m}+\mathfrak{n}]}m_{\mathfrak{m}+\mathfrak{n}}\widehat{\Lambda}_{\mathfrak{m}+\mathfrak{n}}(u; \{u_{\mathfrak{m}+\mathfrak{n}-2}\})\prod_{p=k+1}^{\mathfrak{m}+\mathfrak{n}-1}\widetilde{\mathcal{F}}_p(u), \tag{6.46}$$

where we have introduced

$$\mathcal{F}_\ell(u) = \prod_{j=1}^{M_\ell} \mathfrak{f}_\ell(u^{(\ell+1...\mathfrak{n}+\mathfrak{m}-1)}, u_{\ell j}),$$

$$\widetilde{\mathcal{F}}_\ell(u) = \prod_{j=1}^{M_\ell} \widetilde{\mathfrak{f}}_{\ell+1}(u^{(\ell+1...\mathfrak{n}+\mathfrak{m}-1)}, u_{\ell j}), \qquad \ell \in \{k \dots \mathfrak{m}+\mathfrak{n}-1\},$$

with the convention $u^{(k...l)} = u$ if $k > l$. It remains to compute the values $\widehat{\Lambda}_k(u; \{u_{k-1}\})$:

Lemma 6.2. *The eigenvalue $\widehat{\Lambda}_k(u; \{u_{k-1}\})$ of $\widehat{d}_{kk}(u; \{u_{k-1}\})$ on $\Omega^{(k-1)}$ is given by*

$$\widehat{\Lambda}_k(u; \{u_k\}) = \widehat{\Lambda}_k(u) \prod_{\ell=1}^{k-2} \prod_{j=1}^{M_\ell} \frac{1}{\widetilde{\mathfrak{f}}_{\ell+1}(u^{(\ell+1...k)}, u_{\ell j})}$$

$$= \widehat{\Lambda}_k(u) \prod_{\ell=1}^{k-2} \frac{1}{\widetilde{\mathcal{F}}_\ell(u^{(k+1...\mathfrak{m}+\mathfrak{n}-1)})}, \qquad k = 1, \dots, \mathfrak{m}+\mathfrak{n}, \tag{6.47}$$

where we have used $\widehat{d}_{kk}(u)\Omega = \widehat{\Lambda}_k(u)\Omega$ with

$$\widehat{\Lambda}_k(u) = \Lambda_k(u^{(1...k)}) - \sum_{i=1}^{k-1} q^{2(k-1-i)-4\sum_{l=i+1}^{k-1}[l]} \psi_i((u^{(k)})^{(i)}) \Lambda_i(u^{(1...k)}) \tag{6.48}$$

$$= \mathcal{K}_k(u^{(1...k)}) \lambda_k(u^{(1...k)}) \lambda'_k(u^{(1...k)}). \tag{6.49}$$

Proof. First we introduce a useful property between coproduct and supertrace:

$$\text{str}_a\big(\Delta\big(D_a^{(k)}(u)\big)\big) = \Delta\big(\text{str}_a\big(D_a^{(k)}(u)\big)\big). \tag{6.50}$$

It is obvious because supertrace and coproduct do not act in the same space. We recall the fundamental representation evaluation map for the $\mathcal{A}_{\mathfrak{m}-k+1|\mathfrak{n}}$ algebra:

$$
\begin{array}{rccc}
& \mathcal{A}_{\mathfrak{m}-k+1|\mathfrak{n}} \otimes \text{End}(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}) & \rightarrow & \text{End}(\mathbb{C}^{\mathfrak{m}-k+1|\mathfrak{n}}) \otimes \text{End}(\mathbb{C}^{\mathfrak{m}|\mathfrak{n}}) \\
\pi_a^{(k)}: & T^{(k)}(u) & \mapsto & \overline{\mathbb{R}}_{12}^{(k)}(u, a) \\
& (T^{-1}(\iota(u)))^{(k)} & \mapsto & \mathbb{R}_{21}^{(k)}(u, a).
\end{array}
\tag{6.51}
$$

We also need the representation of $\mathcal{A}_{\mathfrak{m}-k+1|\mathfrak{n}}$ induced by the inclusion $\mathcal{A}_{\mathfrak{m}-k+1|\mathfrak{n}} \hookrightarrow \mathcal{A}_{\mathfrak{m}-p+1|\mathfrak{n}}$, $p < k$. From the identity

$$T^{(k)}(u) = \mathbb{I}^{(k)}(T^{(p)}(u^{(p+1...k)}))\mathbb{I}^{(k)}, \tag{6.52}$$

we can deduce the form of $\pi_v^{(p)}((T^{(k)}(u))$ in the fundamental representation of $\mathcal{A}_{\mathfrak{m}-p+1|\mathfrak{n}}$:

$$\big(\text{id} \otimes \pi_v^{(p)}\big)(T^{(k)}(u)) = \mathbb{I}_1^{(k)}\big(\text{id} \otimes \pi_v^{(p)}\big)(T^{(p)}(u^{(p+1...k)}))\mathbb{I}_1^{(k)} = \mathbb{I}_1^{(k)}\overline{\mathbb{R}}_{12}^{(p)}(u^{(p+1...k)}, v)\mathbb{I}_1^{(k)}$$

$$= \overline{\mathbb{R}}_{a,b}^{(k,p)}(u^{(p+1...k)}, v). \tag{6.53}$$

The last equality is just the definition of $\mathbb{R}_{a,b}^{(k,p)}(u^{(p+1...k)}, v)$, see (2.15).

Hence, using theorem 3.3, we can rewrite the monodromy operator $\widetilde{D}_a^{(k+1)}(u; \{u_{k-1}\})$ as

$$\widetilde{D}_a^{(k+1)}(u; \{u_{k-1}\}) = \left(\overrightarrow{\prod_{j=1}^{M_k}} \overline{\mathbb{R}}_{aa_j^k}^{(k+1)}(u, u_{kj})\right) \widehat{D}_a^{(k+1)}(u; \{u_{k-1}\}) \left(\overleftarrow{\prod_{j=1}^{M_k}} \mathbb{R}_{a_j^k a}^{(k+1)}(u, u_{kj})\right)$$

$$= \big(\text{id} \otimes \big(\pi_{u_{ik}}^{(k+1)}\big)^{\otimes_{i=1}^{M_k}}\big) \circ \Delta^{M_k}\big(\widehat{D}_a^{(k+1)}(u; \{u_{k-1}\})\big), \tag{6.54}$$

27

while the operator $\widetilde{d}_{kk}^{(k)}(u)$ takes the form:

$$\widetilde{d}_{kk}^{(k)}(u;\{u_{k-1}\}) = \left(\mathrm{id} \otimes \left(\left(\pi_{u_{pi}}^{(p+1)}\right)^{\otimes_{i=1}^{M_p}}\right)^{\otimes_{p=1}^{k-1}}\right) \circ \Delta^{\sum_{p=1}^{k} M_p}\left(\widehat{d}_{kk}^{(k)}(u)\right). \qquad (6.55)$$

Now acting on the highest weight $\Omega^{(k-1)}$ and using lemma 3.2, we find the following result:

$$\widetilde{d}_{kk}^{(k)}(u)\Omega^{(k-1)} = \left(\mathrm{id} \otimes \left(\left(\pi_{u_{ip}}^{(p+1)}\right)^{\otimes_{i=1}^{M_p}}\right)^{\otimes_{p=1}^{k}}\right)\left(t_{kk}^{(k)}(u)t_{kk}'^{(k)}(\iota(u))\right)^{\otimes_{p=1}^{k} M_p} \otimes \widehat{d}_{kk}^{(k)}(u)\Omega^{(k-1)}. \tag{6.56}$$

We have from the definition of $\mathbb{R}$ matrices and $\Omega^{(k-1)}$,

$$\pi_{u_{ip}}^{(p+1)}\left(t_{kk}^{(k)}(u)t_{kk}'^{(k)}(\iota(u))\right)\Omega^{(k-1)} = \frac{\mathfrak{b}(u^{(p+1\ldots k)}, u_{ip})\bar{\mathfrak{b}}(u^{(p+1\ldots k)}, u_{ip})}{\mathfrak{a}_{p+1}(u^{(p+1\ldots k)}, u_{ip})\bar{\mathfrak{a}}_{p+1}(u^{(p+1\ldots k)}, u_{ip})}\Omega^{(k-1)},$$
$$\pi_{u_{ik}}^{(k)}\left(t_{kk}^{(k)}(u)t_{kk}'^{(k)}(\iota(u))\right)\Omega^{(k-1)} = \Omega^{(k-1)}, \tag{6.57}$$

that leads to the result (6.47). The eigenvalue (6.48) is computed directly from theorem 3.1. To obtain the form (6.49), one uses the equalities (4.38), (4.39) and the identity (B.10). $\qquad\square$

From the expression given in lemma 6.2, one deduces that

$$\widehat{\Gamma}^{(k+1)}(u^{(k+2\ldots n+m-1)}) = \Bigg\{\widetilde{\mathfrak{m}}_{k+1}(u^{(k+2\ldots n+m-1)})\widehat{\Lambda}_{k+1}(u^{(k+2\ldots n+m-1)})\mathcal{F}_{k+1}(u)$$

$$+ \sum_{\ell=k+2}^{m+n-1} \widetilde{\mathfrak{m}}_\ell(u^{(\ell+1\ldots n+m-1)})\widehat{\Lambda}_\ell(u^{(\ell+1\ldots n+m-1)})\mathcal{F}_l(u)\widetilde{\mathcal{F}}_{l-1}(u)$$

$$+ (-1)^{[m+n]} m_{m+n}\widehat{\Lambda}_{m+n}(u^{(n+m-1)})\widetilde{\mathcal{F}}_{m+n-1}(u)\Bigg\}\frac{1}{\prod_{\ell=1}^{k-1}\widetilde{\mathcal{F}}_\ell(u)}. \tag{6.58}$$

Let us note that since $\mathfrak{b}(u,u) = 0$, equation (6.58) implies that

$$\widehat{\Gamma}^{(k+1)}(u_{\ell j}) = 0, \qquad \text{for} \quad j = 1,\ldots,M_\ell; \quad \ell = 1,\ldots,k-1, \tag{6.59}$$

$$\widehat{\Gamma}^{(k+1)}(u_{kj}) = \widetilde{\mathfrak{m}}_{k+1}(u_{kj})\widehat{\Lambda}_{k+1}(u_{kj})\mathcal{F}_{k+1}\left(u_{kj}^{\overline{(k+2\ldots n+m-1)}}\right)\prod_{\ell=1}^{k-1}\frac{1}{\widetilde{\mathcal{F}}_{\ell+1}\left(u_{kj}^{\overline{(k+2\ldots n+m-1)}}\right)},$$
$$\text{for} \quad j = 1,\ldots,M_k. \tag{6.60}$$

6.5. Final form of Bethe vectors, eigenvalues and equations

Using expressions (6.59), (6.60), and the value of $\widehat{\Lambda}_k(u;\{u^{(k)}\})$ given in lemma 6.2, one can recast the Bethe equations (6.43) in their final form:

$$\frac{\widehat{\Lambda}_k(u_{kj})}{\widehat{\Lambda}_{k+1}\left(u_{kj}^{\overline{(k+1)}}\right)} = \frac{\widetilde{\mathfrak{m}}_{k+1}\left(u_{kj}^{\overline{(k+1)}}\right)\chi_k(u_{kj})}{\mathfrak{e}_{k+1}(u_{kj})}\prod_{i=1}^{M_{k-1}}\frac{1}{\widetilde{\mathfrak{f}}_k\left(u_{kj}^{(k)}, u_{k-1,i}\right)}\prod_{i\neq j}^{M_k}\frac{\widetilde{\mathfrak{f}}_{k+1}(u_{kj}, u_{ki})}{\mathfrak{f}_k(u_{kj}, u_{ki})}$$

$$\times \prod_{i=1}^{M_{k+1}}\mathfrak{f}_{k+1}\left(u_{kj}^{\overline{(k+1)}}, u_{k+1,i}\right), \quad j = 1,\ldots,M_k, \quad k = 1,\ldots,m+n-1, \tag{6.61}$$

with the convention $M_0 = M_{m+n} = 0$. The number of parameter families is $m + n - 1$. We checked that using the weights (4.3), (4.5) and the functions given in appendix B, one reproduces the BAEs already computed, in e.g. [6, 12, 17], and also the general forms given in

28

[13–15]. In particular, for the fundamental weight $\mu = (1, 0, \ldots, 0)$, we recovers the BAEs for a spin chain with fundamental representations. For instance, for the case of $\widehat{\mathcal{U}}_q(2|2)$, which may be of some relevance in the context of AdS/CFT correspondence, one obtains (for L sites with evaluation parameter b_i, $a_+ = 1$ and $a_- = 2$)

$$\frac{c_+ u_{1i} q^{\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{c_+ u_{1i}}}{\frac{u_{1i}}{c_+} q^{\frac{1}{2}} - \frac{c_+}{u_{1i}} q^{-\frac{1}{2}}} \prod_{i=1}^{L} \frac{\left(\frac{u_{1i}}{b_i} q^{\frac{1}{2}} - \frac{b_i}{u_{1i}} q^{-\frac{1}{2}}\right)\left(u_{1i} b_i q^{\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{b_i u_{1i}}\right)}{\left(\frac{u_{1i}}{b_i} q^{-\frac{1}{2}} - \frac{b_i}{u_{1i}} q^{\frac{1}{2}}\right)\left(u_{1i} b_i q^{-\frac{1}{2}} - \frac{q^{\frac{1}{2}}}{b_i u_{1i}}\right)}$$

$$= \prod_{j \neq i}^{M_1} \frac{\left(\frac{u_{1i}}{u_{1j}} q - \frac{u_{1j}}{u_{1i}} q^{-1}\right)\left(u_{1i} u_{1j} q - \frac{q^{-1}}{u_{1i} u_{1j}}\right)}{\left(\frac{u_{1j}}{u_{1i}} q - \frac{u_{1i}}{u_{1j}} q^{-1}\right)\left(u_{1i} u_{1j} q^{-1} - \frac{q}{u_{1i} u_{1j}}\right)}$$

$$\times \prod_{j=1}^{M_2} \frac{\left(\frac{u_{2j}}{u_{1i}} q^{-\frac{1}{2}} - \frac{u_{1i}}{u_{2j}} q^{\frac{1}{2}}\right)\left(u_{2j} u_{1i} q^{-\frac{1}{2}} - \frac{q^{\frac{1}{2}}}{u_{2j} u_{1i}}\right)}{\left(\frac{u_{2j}}{u_{1i}} q^{-\frac{1}{2}} - \frac{u_{2j}}{u_{1i}} q^{\frac{1}{2}}\right)\left(u_{2j} u_{1i} q^{\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{u_{2j} u_{1i}}\right)}, \qquad i = 1, \ldots, M_1$$

$$\frac{u_{2i}^2 q^{-1} - c_-^2 q}{u_{2i}^{-2} - c_-^2 q^2} (-1)^{M_2+1} = \prod_{j=1}^{M_1} \frac{\left(\frac{u_{2i}}{u_{1j}} q^{-\frac{1}{2}} - \frac{u_{1j}}{u_{2i}} q^{\frac{1}{2}}\right)\left(u_{1j} u_{2i} q^{-\frac{1}{2}} - \frac{q^{\frac{1}{2}}}{u_{1j} u_{2i}}\right)}{\left(\frac{u_{2i}}{u_{1j}} q^{\frac{1}{2}} - \frac{u_{1j}}{u_{2i}} q^{-\frac{1}{2}}\right)\left(u_{1j} u_{2i} q^{\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{u_{1j} u_{2i}}\right)}$$

$$\times \prod_{j=1}^{M_3} \frac{\left(\frac{u_{3j}}{u_{2i}} q^{-\frac{1}{2}} - \frac{u_{2i}}{u_{3j}} q^{\frac{1}{2}}\right)\left(u_{3j} u_{2i} q^{\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{u_{3j} u_{2i}}\right)}{\left(\frac{u_{3j}}{u_{2i}} q^{\frac{1}{2}} - \frac{u_{3j}}{u_{2i}} q^{-\frac{1}{2}}\right)\left(u_{3j} u_{2i} q^{-\frac{1}{2}} - \frac{q^{\frac{1}{2}}}{u_{3j} u_{2i}}\right)} \qquad i = 1, \ldots, M_2$$

$$-1 = \prod_{j=1}^{M_2} \frac{\left(\frac{u_{3i}}{u_{2j}} q^{\frac{1}{2}} - \frac{u_{2j}}{u_{3i}} q^{-\frac{1}{2}}\right)\left(u_{2j} u_{3i} q^{\frac{1}{2}} - \frac{q^{-\frac{1}{2}}}{u_{2j} u_{3i}}\right)}{\left(\frac{u_{3i}}{u_{2j}} q^{-\frac{1}{2}} - \frac{u_{2j}}{u_{3i}} q^{\frac{1}{2}}\right)\left(u_{2j} u_{3i} q^{-\frac{1}{2}} - \frac{q^{\frac{1}{2}}}{u_{2j} u_{3i}}\right)}$$

$$\times \prod_{j \neq i}^{M_3} \frac{\left(\frac{u_{3i}}{u_{3j}} q^{-1} - \frac{u_{3j}}{u_{3i}} q\right)\left(u_{3i} u_{3j} q^{-1} - \frac{q}{u_{3i} u_{3j}}\right)}{\left(\frac{u_{3j}}{u_{3i}} q^{-1} - \frac{u_{3i}}{u_{3j}} q\right)\left(u_{3i} u_{3j} q - \frac{q^{-1}}{u_{3i} u_{3j}}\right)} \qquad i = 1, \ldots, M_3.$$

The transfer matrix eigenvalues are obtained from (6.58), remarking that $\Lambda(u^{(1)}) = \widehat{\Gamma}^{(1)}(u)$:

$$\Lambda(u^{(1\ldots\mathfrak{m}+\mathfrak{n}-1)}) = \sum_{k=1}^{\mathfrak{m}+\mathfrak{n}} \widetilde{\mathfrak{m}}_k(u^{(k+1\ldots\mathfrak{m}+\mathfrak{n}-1)}) \widehat{\Lambda}_k(u^{(k+1\ldots\mathfrak{m}+\mathfrak{n}-1)})$$

$$\times \prod_{j=1}^{M_k} \mathfrak{f}_k(u^{(k+1\ldots\mathfrak{m}+\mathfrak{n}-1)}, u_{kj}) \prod_{j=1}^{M_{k-1}} \widetilde{\mathfrak{f}}_k(u^{(k+1\ldots\mathfrak{m}+\mathfrak{n}-1)}, u_{k-1j}), \qquad (6.62)$$

$$\widetilde{\mathfrak{m}}_{\mathfrak{m}+\mathfrak{n}}(u) = (-1)^{[\mathfrak{m}+\mathfrak{n}]} m_{\mathfrak{m}+\mathfrak{n}}.$$

The Bethe equations (6.61) ensure that $\Lambda(u)$ is analytical, in accordance with the analytical Bethe ansatz. The Bethe vectors take the form:

$$\Phi(\{u\}) = \widehat{B}_{a_1}^{(1)}(u_{11}) \cdots \widehat{B}_{a_{M_1}}^{(1)}(u_{1M_1}) \widehat{F}_{a_1 \ldots a_{M_1}}^{(1)}(\{u\}) \Omega,$$

$$= \widehat{B}_{a_1^1}^{(1)}(u_{11}) \cdots \widehat{B}_{a_{M_1}^1}^{(1)}(u_{M_1}^{(1)}) \widehat{\widetilde{B}}_{a_2^2}^{(2)}(u_{21}) \cdots \widehat{\widetilde{B}}_{a_{M_2}^2}^{(2)}(u_{1M_2}) \cdots \widehat{\widetilde{B}}_{a_M^{\mathfrak{n}+\mathfrak{m}-1}}^{(\mathfrak{n}+\mathfrak{m}-1)}(u_{\mathfrak{n}+\mathfrak{m}-1,M}) \Omega^{(\mathfrak{n}+\mathfrak{m}-1)}.$$

$$(6.63)$$

We recall the notation $M = \sum_{j=1}^{\mathfrak{n}+\mathfrak{m}-1} M_j$, $\Omega^{(k)} = \left(e_1^{(k-1)}\right)^{\otimes M_{k-1}} \otimes \Omega^{(k-1)}$, $\Omega^{(1)} = \Omega$ and the auxiliary spaces are indicated according to remark 6.4.

7. Bethe vectors

We present here a generalization to open spin chains of the recursion and trace formulae for Bethe vectors, obtained in [35, 36] (see also [16]) for closed spin chains. To our knowledge, this presentation for open spin chains is entirely new.

7.1. Recursion formula for Bethe vectors

From expression (6.63), we can extract a recurrent form for the Bethe vectors,

$$\Phi_M^{n+m}(\{u\}) = \widehat{B}_{a_1^1}^{(1)}(u_{11}) \cdots \widehat{B}_{a_{M_1}^1}^{(1)}(u_{1M_1}) \widehat{\Psi}_{\{u_1\}}^{(1)}(\Phi_{M-M_1}^{n+m-1}(\{u_{(>1)}\})), \tag{7.1}$$

$$\widehat{\Psi}_{\{u_1\}}^{(1)} = v^{(2)} \circ (\tau \otimes \pi_{u_{1M_1}}^{(2)} \otimes \cdots \otimes \pi_{u_{11}}^{(2)}) \circ \Delta^{(M_1)}, \tag{7.2}$$

where π_a is the fundamental representation evaluation homomorphism normalized as in (6.51), $v^{(k)}$ is the application of the highest weight vector $e_1^{(k-1)}$:

$$v^{(k)}(X\Omega^{(k-1)}) = X(e_1^{(k-1)})^{\otimes M_{k-1}} \otimes \Omega^{(k-1)} = X\Omega^{(k)}, \tag{7.3}$$

and τ is the morphism

$$\tau: \begin{array}{ccc} \mathfrak{D}_{m-1|n} & \to & \mathfrak{D}_{m|n}/\mathcal{I}_1 \\ d_{ij}(u) & \mapsto & \widehat{d}_{i+1,j+1}(u) \end{array}. \tag{7.4}$$

If we denote by $[.]_{m|n}$ the grading used in the $\mathfrak{D}_{m|n}$ superalgebra, the mapping τ corresponds to the identification $[j]_{m-1|n} = [j+1]_{m|n}$.

Remark that since all the operators in (7.1) apply on the pseudo-vacuum, one can consider the operators built from τ as belonging to $\mathfrak{D}_{m|n}$ instead of $\mathfrak{D}_{m|n}/\mathcal{I}_1$. So by induction we build the Bethe vectors from $\mathfrak{D}_{m|n}$ generators and R-matrices in auxiliary spaces.

7.2. Supertrace formula for Bethe vectors

We can also write the Bethe vector into a supertrace formula and prove the equivalence with the recursion relation discussed above.

Theorem 7.1. *The Bethe vector (7.1) admits a supertrace formulation. We denote by* $A_1; \ldots;$
A_{m+n-1} *the ordered sequence of auxiliary spaces* $a_1^1, \ldots, a_{M_1}^1;$ $a_1^2, \ldots, a_{M_2}^2; \ldots;$
$a_1^{m+n-1}, \ldots, a_{M_{m+n-1}}^{m+n-1}.$

$$\Phi_M^{n+m}(\{u\}) = (-1)^{G_1} \mathrm{str}_{A_1 \ldots A_{m+n-1}} \left(\prod_{i=1}^{m+n-1} \widehat{\mathbb{D}}_{A_i}^{(i)}(\{u_i\}) E_{n+m,n+m-1}^{\bar{\otimes}M_{n+m-1}} \otimes \cdots \otimes E_{21}^{\bar{\otimes}M_1} \right) \Omega, \tag{7.5}$$

where

$$\widehat{\mathbb{D}}_{A_i}^{(i)}(\{u_i\}) = \prod_{j=1}^{M_i} \overline{\mathcal{R}}_{A_{<i},a_j^i}^{(i)}(\{u_{i-1}\}, u_{ij}) \widehat{D}_{a_j^i}^{(i)}(u_{ij}) \mathcal{R}_{a_j^i,A_{<i}}^{(i)}(\{u_{i-1}\}, u_{ij}), \tag{7.6}$$

$$G_k = \sum_{i=k}^{n+m-2} \frac{M_i(M_i+1)}{2}[i], \tag{7.7}$$

$$\overline{\mathcal{R}}_{A_{<i},a_j^i}^{(i)}(\{u_{i-1}\}, u_{ij}) = \overrightarrow{\prod_{b<i}} \overrightarrow{\prod_{c=1}^{M_b}} \overline{\mathbb{R}}_{a_j^i a_c^b}^{(i,b)}(u_{ij}, u_{bc}^{(b+1\ldots i-1)}), \tag{7.8}$$

$$\mathcal{R}^{(i)}_{a_j^i, A_{<i}}(\{u_{i-1}\}, u_{ij}) = \prod_{b<i} \overset{\overleftarrow{M_b}}{\prod_{c=1}} \mathbb{R}^{(b,i)}_{a_c^b a_j^i}\big(u_{bc}^{(b+1\ldots i-1)}, u_{ij}\big). \tag{7.9}$$

Proof. Equivalence is proven along the following lines. Starting from expression (7.5), we can extract the M_1 auxiliary spaces corresponding to the first step of the nested Bethe ansatz:

$$\Phi_M^{\mathfrak{n}+\mathfrak{m}}(\{u\}) = (-1)^{\frac{M_1(M_1+1)}{2}[1]} \operatorname{str}_{A_1}\Bigg[\widehat{\mathbb{D}}^{(1)}_{A_1}(\{u_1\}) \times (-1)^{G_2}$$

$$\times \operatorname{str}_{A_2\ldots A_{\mathfrak{m}+\mathfrak{n}-1}}\Bigg(\prod_{i=2}^{\mathfrak{m}+\mathfrak{n}-1} \widehat{\mathbb{D}}^{(i)}_{A_i}(\{u_i\}) E^{\vec{\otimes}M_{\mathfrak{n}+\mathfrak{m}-1}}_{\mathfrak{n}+\mathfrak{m},\mathfrak{n}+\mathfrak{m}-1} \otimes \cdots \otimes E^{\vec{\otimes}M_2}_{32} \Bigg) \otimes E^{\vec{\otimes}M_1}_{21} \Bigg] \otimes \Omega.$$

Using the isomorphism $\operatorname{End}(\mathbb{C}^{\mathfrak{m}+\mathfrak{n}}) \sim \mathbb{C}^{\mathfrak{m}+\mathfrak{n}} \otimes \mathbb{C}^{\mathfrak{m}+\mathfrak{n}}$, one can rewrite, for any $A(v)$, the supertrace with an E_{21} matrix as

$$\operatorname{str}(\widehat{D}^{(1)}(u) A(v) E_{21}) = \sum_{j=1}^{\mathfrak{m}+\mathfrak{n}} \big(e_1^t \otimes e_j^t \otimes \widehat{d}^{(1)}_{1j}(u) \big) A(v)(e_1 \otimes e_2 \otimes 1),$$

$$= (-1)^{[1]+[1][A]} \widehat{B}^{(1)}(u) A(v)(e_2 \otimes 1). \tag{7.10}$$

Using formula (7.10) for the auxiliary spaces $1, \ldots, M_1$, and remarking that the case $j_a = 1$ for $a = 1, \ldots, M_1$ does not contribute, we obtain

$$\Phi_M^{\mathfrak{n}+\mathfrak{m}}(\{u\}) = \widehat{B}^{(1)}_{a_1^1}(u_{11}) \cdots \widehat{B}^{(1)}_{a_{M_1}^1}(u_{1M_1})(-1)^{G_2}$$

$$\times \operatorname{str}_{A_2\ldots A_{\mathfrak{m}+\mathfrak{n}-1}}\Bigg(\prod_{i=2}^{\mathfrak{m}+\mathfrak{n}-1} \widehat{\mathbb{D}}^{(i)}_{A_i}(\{u_i\}) E^{\vec{\otimes}M_{\mathfrak{n}+\mathfrak{m}-1}}_{\mathfrak{n}+\mathfrak{m},\mathfrak{n}+\mathfrak{m}-1} \otimes \cdots \otimes E^{\vec{\otimes}M_2}_{32} \Bigg) \Omega^{(2)}. \tag{7.11}$$

To end the proof, we remark that

$$(-1)^{G_2} \operatorname{str}_{A_2\ldots A_{\mathfrak{m}+\mathfrak{n}-1}}\Bigg(\prod_{i=2}^{\mathfrak{m}+\mathfrak{n}-1} \widehat{\mathbb{D}}^{(i)}_{A_i}(\{u_i\}) E^{\vec{\otimes}M_{\mathfrak{n}+\mathfrak{m}-1}}_{\mathfrak{n}+\mathfrak{m},\mathfrak{n}+\mathfrak{m}-1} \otimes \cdots \otimes E^{\vec{\otimes}M_2}_{32} \Bigg) \Omega^{(2)}$$

$$= \widehat{\Psi}^{(1)}_{\{u_1\}}\big(\Phi_{M-M_1}^{\mathfrak{n}+\mathfrak{m}-1}(\{u_{(>1)}\}) \big)$$

which allows us to recover the form (7.2). $\qquad\square$

Remark 7.1 (conjecture). Although theorem 7.1 has been proven only when $K^+(u)$ belongs to a NABA couple, expression (7.5) does not depend on $K^+(u)$: we conjecture that this expression is valid for any couple of diagonal $K^\pm(u)$ matrices. This conjecture is supported by the fact that the analytical Bethe ansatz is known to work for any diagonal boundary matrices.

7.3. Examples of Bethe vectors

To illustrate the supertrace formula, we present here some explicit examples of Bethe vectors associated with small numbers of excitations.

Bethe vectors of $\mathcal{D}_{\mathfrak{m}|\mathfrak{n}}$ with $\mathfrak{n} + \mathfrak{m} = 2$ and $M_1 = M$. We reproduce here the well-known case obtained with algebraic Bethe ansatz (see also section 5).

$$\Phi_M^2(\{u\}) = (-1)^{M[2]} \widehat{d}^{(1)}_{12}(u_{11}) \cdots \widehat{d}^{(1)}_{12}(u_{1M}) \Omega. \tag{7.12}$$

Note that this expression is also valid when $\mathfrak{n} + \mathfrak{m} > 2$, setting $M_1 = M$ and $M_k = 0, k > 1$.

Bethe vectors of $\mathcal{D}_{\mathfrak{m}|\mathfrak{n}}$ with $\mathfrak{n} + \mathfrak{m} = 3$, $M_1 = 1$ and $M_2 = 1$.

$$\Phi_{1,1}^3(\{u\}) = (-1)^{[1]+[2]+[3]} \frac{\mathfrak{b}(u_{11}, u_{21})}{\mathfrak{a}_2(u_{11}, u_{21})} \widehat{d}_{12}^{(1)}(u_{11}) \widehat{d}_{23}^{(2)}(u_{21}) \Omega$$

$$+ (-1)^{[1]+[2]+[3]} \frac{\overline{\mathfrak{b}}(u_{21}, u_{11})}{\overline{\mathfrak{a}}_2(u_{21}, u_{11})} \frac{\mathfrak{w}_{32}(u_{11}, u_{21})}{\mathfrak{a}_2(u_{11}, u_{21})} \widehat{d}_{13}^{(1)}(u_{11}) \widehat{d}_{22}^{(2)}(u_{21}) \Omega$$

$$+ (-1)^{[1]} \frac{\overline{\mathfrak{w}}_{23}(u_{21}, u_{11})}{\overline{\mathfrak{a}}_2(u_{21}, u_{11})} \frac{\mathfrak{b}(u_{11}, u_{21})}{\mathfrak{a}_2(u_{11}, u_{21})} \widehat{d}_{13}^{(1)}(u_{11}) \widehat{d}_{33}^{(2)}(u_{21}) \Omega.$$

Again, this expression is also valid when $\mathfrak{n} + \mathfrak{m} > 3$, setting $M_k = 0$, $k > 2$.

We also computed the Bethe vectors corresponding to $M_1 = M_2 = M_3 = 1$ and $M_k = 0$, $k > 3$. Their expression is rather long, with 11 different terms: we do not write it here explicitly.

8. Conclusion

In this paper, we have proposed a global treatment of the NBA for *universal transfer matrices* of open spin chains with *NABA couple* of boundary matrices. The modification of the nested Bethe ansatz applicable to diagonal boundary matrices that do not form a NABA couple remains to be found. Since the analytical Bethe ansatz can be performed in this case, such a refinement should be possible.

We have computed a *trace formula* for the Bethe vector of the open chain. This formulation could be a starting point for the investigation of the quantized Knizhnik–Zamolodchikov equation following the work [37]. For such a purpose, the coproduct properties of Bethe vectors for open spin chains remain to be studied. Defining a scalar product and computing the norm of these Bethe vectors is also a point of fundamental interest.

From a different point of view, this trace formula and the mapping between the reflection algebras of different sizes could be the starting point for the construction of a Drinfeld's current realization [21] for the reflection algebra in the spirit of [38] on the current realization of the Bethe vector for the periodic case.

The case of open spin chains with general boundary matrices is also a subject of fundamental interest. A deeper understanding of representations of reflection algebras when the K matrix is not diagonal may be of some help. Alternatively, a different approach using another presentation of the reflection algebra could be the clue to go beyond the results obtained so far. Some works have been done for the $\mathfrak{m} + \mathfrak{n} = 2$ case in [7], but the general treatment for universal transfer matrices remains an open problem. The functional approach developed in [8] for $\mathfrak{m} + \mathfrak{n} = 2$ also deserves a generalization, both for universal transfer matrices, and for bigger algebras.

Appendix A. R and M matrices

We recall the general form of the R-matrices we used in the paper [16]. Note we use a more compact form:

$$R_{12}(u, v) = \mathfrak{b}(u, v) \mathbb{I} \otimes \mathbb{I} + \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} \mathfrak{w}_{ij}(u, v) E_{ij} \otimes E_{ji}, \tag{A.1}$$

$$\mathfrak{w}_{ij}(u, v) = \begin{cases} \mathfrak{a}_i(u, v) - \mathfrak{b}(u, v) & \text{for} \quad i = j \\ \mathfrak{c}_{ij}(u, v) & \text{otherwise,} \end{cases}$$

$$\bar{R}_{12}(u, v) = R_{12}(u, \iota(v)) = \bar{\mathfrak{b}}(u, v)\mathbb{I} \otimes \mathbb{I} + \sum_{i,j=1}^{\mathfrak{m}+\mathfrak{n}} \bar{\mathfrak{w}}_{ij}(u, v) E_{ij} \otimes E_{ji}. \quad (A.2)$$

The functions involved in these expressions are given by (with the convention $Y(\mathfrak{m}) \equiv Y(\mathfrak{m}|0)$, $\widehat{\mathcal{U}}_q(\mathfrak{m}) \equiv \widehat{\mathcal{U}}_q(\mathfrak{m}|0)$ and $[b] = 0$, $\forall b$, in these two cases):

For $Y(\mathfrak{m}|\mathfrak{n})$:

$$\mathfrak{b}(u, v) = u - v; \qquad \mathfrak{a}_a(u, v) = u - v - (-1)^{[a]}\hbar \quad \text{and} \quad \mathfrak{w}_{ab}(u, v) = -(-1)^{[b]}\hbar \quad (A.3)$$

$$\bar{\mathfrak{b}}(u, v) = u + v; \qquad \bar{\mathfrak{a}}_a(u, v) = u + v - (-1)^{[a]}\hbar \quad \text{and} \quad \bar{\mathfrak{w}}_{ab}(u, v) = -(-1)^{[b]}\hbar. \quad (A.4)$$

For $\widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$:

$$\mathfrak{b}(u, v) = \frac{u}{v} - \frac{v}{u}; \qquad \mathfrak{a}_a(u, v) = \frac{u}{v}q^{1-2[a]} - \frac{v}{u}q^{2[a]-1} \quad (A.5)$$

$$\text{and} \quad \mathfrak{w}_{ab}(u, v) = (-1)^{[b]}(q - q^{-1})\left(\frac{u}{v}\right)^{\text{sign}(b-a)}, \qquad a \neq b \quad (A.6)$$

$$\bar{\mathfrak{b}}(u, v) = uv - \frac{1}{uv}; \qquad \bar{\mathfrak{a}}_a(u, v) = uvq^{1-2[a]} - \frac{1}{uv}q^{2[a]-1} \quad (A.7)$$

$$\text{and} \quad \bar{\mathfrak{w}}_{ab}(u, v) = (-1)^{[b]}(q - q^{-1})(uv)^{\text{sign}(b-a)}, \qquad a \neq b. \quad (A.8)$$

The matrix M is a diagonal matrix:

$$M = \sum_{i=1}^{\mathfrak{m}+\mathfrak{n}} m_i E_{ii} \qquad \text{with} \quad \begin{cases} m_i = 1 & \text{for} \quad Y(\mathfrak{m}|\mathfrak{n}) \\ m_i = q^{\mathfrak{m}-\mathfrak{n}-2k+1}q^{-2[k]+4\sum_{i=1}^{k}[i]} & \text{for} \quad \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n}). \end{cases} \quad (A.9)$$

Appendix B. Functions appearing in NBA

The functions are constructed from the three functions appearing in the R-matrix, whose explicit forms are given in equations (A.3)–(A.8) above:

$$\mathfrak{f}_i(u, v) = \frac{\mathfrak{a}_i(v, u)\bar{\mathfrak{b}}(u^{(i)}, v^{(i)})}{\mathfrak{b}(v, u)\bar{\mathfrak{b}}(u, v)}, \qquad \widetilde{\mathfrak{f}}_{i+1}(u, v) = \frac{\mathfrak{a}_{i+1}(u, v)\bar{\mathfrak{a}}_{i+1}(u, v)}{\mathfrak{b}(u, v)\bar{\mathfrak{b}}(u, v)}$$

$$\mathfrak{g}_i(u, v) = \frac{\mathfrak{c}_{i-1\,i}(u, v)\bar{\mathfrak{b}}(v^{(i)}, v^{(i)})}{\mathfrak{b}(u, v)\bar{\mathfrak{b}}(v, v)}, \quad \widetilde{\mathfrak{g}}_{i+1}(u, v) = -(-1)^{[i]+[i+1]}\frac{\mathfrak{c}_{k+1\,k}(u, v)\bar{\mathfrak{a}}_{k+1}(u, u)}{\mathfrak{b}(u, v)\bar{\mathfrak{b}}(u, u)} \quad (B.1)$$

$$\mathfrak{h}_i(u, v) = -\frac{\bar{\mathfrak{c}}_{i-1\,i}(u^{(i)}, v^{(i)})}{\bar{\mathfrak{b}}(u, v)}, \qquad \widetilde{\mathfrak{h}}_{i+1}(u, v) = \frac{\bar{\mathfrak{c}}_{i+1\,i}(u^{(i)}, v^{(i)})\bar{\mathfrak{a}}_{i+1}(u, u)\bar{\mathfrak{b}}(v^{(i)}, v^{(i)})}{\bar{\mathfrak{b}}(u, u)\bar{\mathfrak{b}}(v, v)\bar{\mathfrak{b}}(u, v)}.$$

We also use (presented here for $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}} = \widehat{\mathcal{U}}_q(\mathfrak{m}|\mathfrak{n})$; for $\mathcal{A}_{\mathfrak{m}|\mathfrak{n}} = Y(\mathfrak{m}|\mathfrak{n})$ one has to set $q = 1$ in the relations below)

$$\psi_i(u) = \frac{\bar{\mathfrak{c}}_{i+1\,i}(u, u)}{\bar{\mathfrak{a}}_i(u, u)} \quad (B.2)$$

$$\chi_k(u) = \begin{cases} -q^{2[k]-1}\dfrac{\bar{\mathfrak{b}}(u, u)}{\bar{\mathfrak{b}}(u^{(k)}, u^{(k)})}\eta_k(u, c_+) & \text{for} \quad k = 1 \quad \text{and} \quad a = 1 \\ -q^{2[k]-1}\dfrac{\bar{\mathfrak{b}}(u, u)}{\bar{\mathfrak{b}}(u^{(k)}, u^{(k)})} & \text{else} \end{cases} \quad (B.3)$$

33

$$\eta_k(u, c_+) = \frac{\mathfrak{b}(\widetilde{c}_+, u^{(k+1)})}{\overline{\mathfrak{b}}(\widetilde{c}_+, \overline{u^{(k+1)}})} \tag{B.4}$$

$$\mathfrak{e}_k(u)\mathbb{I}_b^{(k)} = \mathrm{tr}_a\big(M_a^{(k)}\bar{\mathbb{R}}_{ab}^{(k)}(u, u)\mathbb{R}_{ba}^{(k)}(u, u)\big)$$

$$= q^{-k+1}q^{-2\mathfrak{n}-2\sum_{i=k}^{\mathfrak{m}+\mathfrak{n}}[i]+4\sum_{i=1}^{\mathfrak{m}+\mathfrak{n}}[i]}(-1)^{[k]}\frac{\overline{\mathfrak{b}}\big(u^{\overline{(k...\mathfrak{m}+\mathfrak{n})}}, u^{\overline{(k...\mathfrak{m}+\mathfrak{n})}}\big)}{\bar{\mathfrak{a}}_k(u, u)}\mathbb{I}_b^{(k)} \tag{B.5}$$

$$\widetilde{\mathfrak{m}}_k(u) = q^{1-2[k]}\frac{\bar{\mathfrak{a}}_{k+1}(u, u)\mathfrak{e}_{k+1}(u)}{\overline{\mathfrak{b}}(u, u)} \qquad \text{for} \quad k \neq 1. \tag{B.6}$$

The following useful relations are used in the paper:

$$\mathfrak{b}(u, v)\mathfrak{a}_i(u^{\overline{(j)}}, v^{\overline{(j)}}) = \mathfrak{a}_i(u, v)\mathfrak{a}_j(u, v) - \mathfrak{w}_{ij}(u, v)\mathfrak{w}_{ji}(u, v), \qquad i \neq j, \tag{B.7}$$

$$\mathfrak{b}(u, v)\mathfrak{w}_{ij}(u^{\overline{(k)}}, v^{\overline{(k)}}) = \mathfrak{w}_{ij}(u, v)\mathfrak{a}_k(u, v) - \mathfrak{w}_{ik}(u, v)\mathfrak{w}_{kj}(u, v), \qquad i > j > k,$$

$$\mathfrak{b}(u^{\overline{(i)}}, v^{\overline{(i)}}) = \mathfrak{a}_i(u, v), \tag{B.8}$$

$$\mathfrak{a}_j(u, v) - \mathfrak{w}_{ij}(u, v) = \mathfrak{b}(u, v)q^{siq(j-i)(-1+2[j])}, \tag{B.9}$$

$$q^{2-4[i]}\psi_{i-1}(u^{(i-1)}) = \psi_{i-1}(u^{(i-1,i)}) - \psi_{i-1}(u^{(i-1,i)})\psi_i(u^{(i)}). \tag{B.10}$$

References

[1] Cherednik I 1984 Factorizing particles on the half-line and roots systems *Teor. Mat. Fiz.* **61** 55
[2] Sklyanin E K 1988 Boundary conditions for integrable quantum systems *J. Phys. A: Math. Gen.* **21** 2375
[3] Faddeev L 1998 How algebraic Bethe ansatz works for integrable model in Symétries quantiques *Les Houches Summer School Proceedings* vol 64, ed A Connes, K Gawedzki and J Zinn-Justin (Amsterdam: North-Holland) (arXiv:hep-th/9605187)
[4] Takhtajan L A and Faddeev L D 1979 The quantum method of the inverse problem and the Heisenberg XYZ model *Russ. Math. Surv.* **34** 11–68
[5] Kulish P P and Sklyanin E K 1979 Quantum inverse scattering method and the Heisenberg ferromagnet *Phys. Lett. A* **70** 461–3
　　Kulish P P and Sklyanin E K 1982 Solution of the Yang-Baxter equation *J. Sov. Math.* **19** 1596–620
[6] Kulish P P and Yu Reshetikhin N 1983 Diagonalisation of $gl(\mathfrak{n})$ invariant transfer matrices and quantum *N*-wave system (Lee model) *J. Phys. A: Math. Gen.* **16** L591–6
[7] Baseilhac P and Koizumi K 2007 Exact spectrum of the XXZ open spin chain from the q-onsager algebra representation theory *J. Stat. Mech.* P09006 (arXiv:hep-th/0703106)
　　Baseilhac P 2007 New results in the XXZ open spin chain *Proc. RAQIS07 (Annecy-le-Vieux, France)* (arXiv:0712.0452)
[8] Kuniba A, Sakai K and Suzuki J 1998 Continued fraction TBA and functional relations in XXZ model at root of unity *Nucl. Phys.* B **525** 597 (arXiv:math/9803056)
　　Nepomechie R 2002 Solving the open XXZ spin chain with nondiagonal boundary terms at roots of unity *Nucl. Phys.* B **622** 615 (arXiv:hep-th/0110116)
　　Nepomechie R 2004 Bethe Ansatz solution of the open XXZ chain with nondiagonal boundary terms *J. Phys. A: Math. Gen.* **37** 433 (arXiv:hep-th/0304092)
　　Murgan R, Nepomechie R and Shi Chi 2006 Exact solution of the open XXZ chain with general integrable boundary terms at roots of unity *J. Stat. Mech.* P08006 (arXiv:hep-th/0605223)
　　Frappat L, Nepomechie R and Ragoucy E 2007 Complete Bethe Ansatz solution of the open spin-s XXZ chain with general integrable boundary terms *J. Stat. Mech.* P09008 (arXiv:0707.0653)
[9] Cao Junpeng, Lin H-Q, Shi K-j and Wang Y 2003 Exact solutions and elementary excitations in the XXZ spin chain with unparallel boundary fields *Nucl. Phys.* B **663** 487 (arXiv:cond-mat/0212163)
[10] Galleas W and Martins M J 2005 Solution of the $SU(\mathfrak{n})$ vertex model with non-diagonal open boundaries *Phys. Lett. A* **335** 167 (arXiv:nlin.SI/0407027)
[11] Yang W L and Zhang Y Z 2005 Exact solution of the $A_{\mathfrak{n}-1}^{(1)}$ trigonometric vertex model with non-diagonal open boundaries *J. High Energy Phys.* JHEP01(2005)021 (arXiv:hep-th/0411190) JHEP01(2005)021 (arXiv:hep-th/0411190)

[12] Yu Reshetikhin N 1987 The spectrum of the transfer matrices connected with Kac-Moody algebras *Lett. Math. Phys.* **14** 235

[13] Arnaudon D, Crampé N, Doikou A, Frappat L and Ragoucy É 2005 Analytical Bethe Ansatz for closed and open $gl(\mathfrak{n})$-spin chains in any representation *J. Stat. Mech.* P02007 (arXiv:math-ph/0411021)

[14] Arnaudon D, Crampé N, Doikou A, Frappat L and Ragoucy É 2006 Spectrum and Bethe ansatz equations for the $U_q(gl(\mathfrak{n}))$ closed and open spin chains in any representation *Ann. H. Poincaré* **7** 1217 (arXiv: math-ph/0512037)

[15] Ragoucy É and Satta G 2007 Analytical Bethe ansatz for closed and open $gl(\mathfrak{m}|\mathfrak{n})$ super-spin chains in arbitrary representations *J. High Energy Phys.* JHEP09(2007)001 (arXiv:hep-th/0706.3327) JHEP09(2007)001 (arXiv:hep-th/0706.3327)

[16] Belliard S and Ragoucy É 2008 The nested Bethe ansatz for 'all' closed spin chains *J. Phys. A: Math. Theor.* **41** 295202 (arXiv:0804.2822 [math-ph])

[17] de Vega H J and González-Ruiz A 1993 Boundary K-matrices for the six vertex and the $n(2n-1)A_{n-1}$ vertex models *J. Phys. A: Math. Gen.* **26** L519 (arXiv:hep-th/9211114)
de Vega H J and González-Ruiz A 1994 Exact solution of the $SU_q(n)$ invariant quantum spin chains *Nucl. Phys.* B **417** 553 (arXiv:hep-th/9309022)
de Vega H J and González-Ruiz A 1994 Exact Bethe ansatz solution of $A_{\mathfrak{n}-1}$ chains with non-$SU_q(\mathfrak{n})$ invariant open boundary conditions arXiv:hep-th/9404141

[18] Yue R H, Fan H and Hou B 1996 Exact diagonalisation of the quantum supersymmetric $SU_q(\mathfrak{n}|\mathfrak{m})$ model *Nucl. Phys.* B **462** 167 (arXiv:cond-mat/9603022)

[19] Yu Reshetikhin N, Takhtajan L A and Faddeev L D 1990 Quantization of Lie Groups and Lie algebras *Leningr. Math. J.* **1** 193–225

[20] Drinfeld V G 1987 Quantum groups *Proc. Int. Congress Math. (Berkeley, 1986)* (Providence, RI: AMS) pp 798–820

[21] Drinfeld V G 1988 A new realization of Yangians and quantized affine algebras *Sov. Math. Dokl.* **36** 212–6

[22] Jimbo M 1985 A q difference analog of U(G) and the Yang–Baxter equation *Lett. Math. Phys.* **10** 63
Jimbo M 1986 A q-difference analogue of $U(gl(n+1))$, Hecke algebra and the Yang–Baxter equation *Lett. Math. Phys.* **11** 247

[23] Reshetikhin N and Semenov-Tian-Shansky M 1990 Central extensions of quantum current groups *Lett. Math. Phys.* **19** 133

[24] Arnaudon D, Avan J, Crampé N, Doikou A, Frappat L and Ragoucy É 2004 General boundary conditions for the $sl(\mathfrak{n})$ and $sl(\mathfrak{m}|\mathfrak{n})$ open spin chains *J. Stat. Mech.* P08005 (arXiv:math-ph/0406021)

[25] Gandenberger G M 1999 New non-diagonal solutions to the $A_n^{(1)}$ boundary Yang–Baxter equation arXiv: hep-th/9911178

[26] Molev A and Ragoucy É 2002 Representations of reflection algebras *Rev. Math. Phys.* **14** 317 (arXiv: math-QA/0406021)

[27] Chari V and Pressley A 1990 Yangians and R-matrices *L'Enseignement Math.* **36** 267
Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[28] Molev A I 2002 Irreducibility criterion for tensor products of Yangian evaluation modules *Duke Math. J.* **112** 307 (arXiv:math.QA/0009183)

[29] Zhang R B 1996 The $gl(\mathfrak{m}|\mathfrak{n})$ super-Yangian and its finite dimensional representations *Lett. Math. Phys.* **37** 419 (arXiv:q-alg/9507029)
Zhang R B 1993 Finite dimensional irreducible representations of the supergroup $U_q(gl(\mathfrak{m}|\mathfrak{n}))$ *J. Math. Phys.* **34** 1236

[30] Rosso M 1988 Finite dimensional representations of the quantum analogue of the enveloping algebra of a complex simple Lie algebra *Commun. Math. Phys.* **117** 581

[31] Molev A I 2003 Yangians and their applications *Handbook of Algebra* vol 3, ed M Hazewinkel (Amsterdam: Elsevier) p 907 (arXiv:math.QA/021128)

[32] Nazarov M 1991 Quantum Berezinian and the classical Capelli identity *Lett. Math. Phys.* **21** 123

[33] Belliard S and Ragoucy É, Berezinian for $\hat{U}_q(gl(\mathfrak{m}|\mathfrak{n}))$ in preparation

[34] Sklyanin E, Takhtajan L and Faddeev L 1979 The quantum inverse problem method *Theor. Math. Phys.* **40** 688

[35] Mukhin E, Tarasov V and Varchenko A 2006 Bethe eigenvectors of higher transfer matrices *J. Stat. Mech.* P08002 (arXiv:math.QA/0605015)

[36] Tarasov V and Varchenko A 2007 Combinatorial formulae for nested Bethe vector arXiv:math.QA/0702277

[37] Tarasov V and Varchenko A 1994 Solutions to the quantized Knizhnik-Zamolodchikov equation and the Bethe ansatz arXiv:hep-th/9411181

[38] Khoroshkin S and Pakuliak S 2005 Weight function for $U_q(\widehat{\mathfrak{sl}}_3)$ *Theor. Math. Phys.* **145** 1373
Khoroshkin S and Pakuliak S 2007 A computation of an universal weight function for the quantum affine algebra $U_q(\widehat{gl}_N)$ arXiv:0711.2819 [math.QA]